Weakly maximal subgroups of branch groups

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Slides available at
`www.leemann.website/slides/weaklymaximal.pdf`

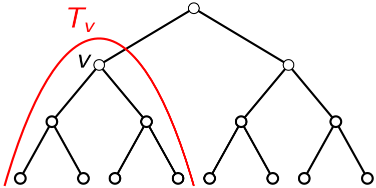
Main goals

- Describe two families of weakly maximal subgroups of branch groups;
- Give several characterizations of these families;
- Show that every weakly maximal subgroup of the first Grigorchuk group belongs to one of these two families.

Introduction

Aut(T_d)

- $T = T_d$: the d-regular rooted tree (the root has degree d and each other vertex has degree $d + 1$);

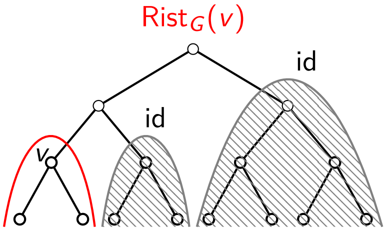


- Vertices of T_d are in bijection with finite words on the alphabet $\{0, \ldots, d-1\}$ (root $\leftrightarrow \emptyset$ the empty word);
- Aut(T_d) is the automorphism group of T_d;
- T_v is the subtree of T consisting of vertices below v.

Some subgroups of Aut(T_d)

Let $G \leq \mathrm{Aut}(T_d)$. The following subgroups play an important role.

- Stabilizers of vertices $\mathrm{Stab}_G(v)$ and of rays $\mathrm{Stab}_G(\xi)$, $\xi \in \partial T$;
- Pointwise stabilizers of levels $\mathrm{Stab}_G(\mathcal{L}_n)$;
- Rigid stabilizer of vertices:
  $\mathrm{Rist}_G(v) := \{g \in G \mid g \text{ acts trivially outside } T_v\}$;

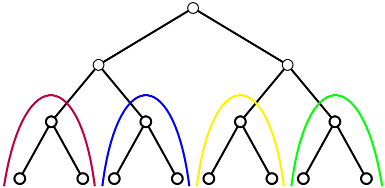


Some subgroups of $\mathrm{Aut}(T_d)$

Let $G \leq \mathrm{Aut}(T_d)$. The following subgroups play an important role.

- Stabilizers of vertices $\mathrm{Stab}_G(v)$ and of rays $\mathrm{Stab}_G(\xi)$, $\xi \in \partial T$;
- Pointwise stabilizers of levels $\mathrm{Stab}_G(\mathcal{L}_n)$;
- Rigid stabilizer of vertices;
- Rigid stabilizer of levels: $\mathrm{Rist}_G(\mathcal{L}_n) := \prod_{v \in \mathcal{L}_n} \mathrm{Rist}_G(v)$.

$$\mathrm{Rist}_G(\mathcal{L}_2) = \{g \in G \mid G = (g_0, g_1, g_2, g_3)\}$$



Branch groups

Definition

A subgroup G of $\mathrm{Aut}(T_d)$ is branch if all the $\mathrm{Rist}_G(\mathcal{L}_n)$ are finite index subgroups of G and G acts minimally on ∂T.

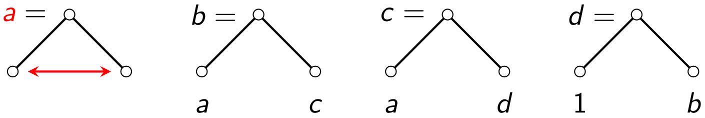
Example

The first Grigorchuk group $\mathfrak{G}$, the Gupta-Sidki p-groups ($p \geq 3$ prime), torsion GGS groups (acting on T_p, $p \geq 3$ prime).

All these examples are infinite, just infinite, torsion, of finite rank, all their maximal subgroups are of finite index, ...

The first Grigorchuk group

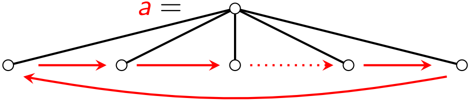
The first Grigorchuk group $\mathfrak{G} = \langle a, b, c, d \rangle$ acts on T_2 and is generated by



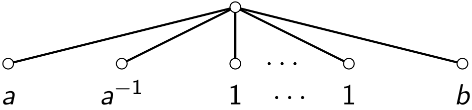
The Gupta-Sidki p-group

The group G_p acts on T_p ($p \geq 3$ prime) and is generated by a and b, where

$a =$



$b =$



GGS groups

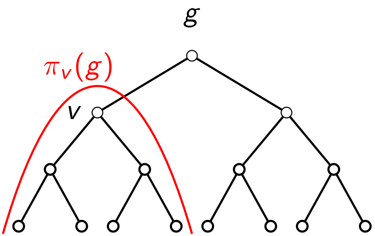
- Let $p \geq 3$ be a prime and let $\mathbf{e} = (e_0, \dots, e_{p-2})$ be a vector in $(F_p)^{p-1} \setminus \{0\}$. The GGS group $G_{\mathbf{e}} = \langle a, b \rangle$ with defining vector $\mathbf{e}$ is the subgroup of $\mathrm{Aut}(T_p)$ generated by

$$a = \text{cyclic permutation } (12 \dots p) \text{ of the first level vertices}$$
$$b = (a^{e_0}, \dots, a^{e_{p-2}}, b)$$

- The group $G_{\mathbf{e}}$ is torsion if and only if $\sum_{i=0}^{p-2} e_i = 0$.
- The Gupta-Sidki p-group correspond to the special case $\mathbf{e} = (1, -1, 0, \dots, 0)$.

Self-replicating groups

For v a vertex of T and $g \in \mathrm{Stab}_{\mathrm{Aut}(T)}(v)$, the section $\pi_v(g)$ of g at v is the automorphism of T_v induced by g.



Definition

A group $G \leq \mathrm{Aut}(T)$ is self-replicating (or fractal) if for every vertex v in T we have $\pi_v(\mathrm{Stab}_G(v)) = G$.

Example

The groups $\mathfrak{G}$ and $G_{\mathbf{e}}$ are self-replicating.

Weakly maximal subgroups

Recall that a maximal subgroup of G is a maximal element in the lattice of proper subgroups of G.

Definition

A **weakly maximal subgroup** is a maximal element in the lattice of infinite index subgroups of G.

Weakly maximal subgroups

- If G is finitely generated, then every infinite index subgroup is contained in a weakly maximal subgroup (use Zorn's Lemma).
- If $M \leq G$ is both maximal and of infinite index, then it is weakly maximal.
- If $G \leq \mathrm{Aut}(T)$ is branch, then the parabolic subgroups $\mathrm{Stab}_G(\xi)$, $\xi \in \partial T$, are weakly maximal, infinite and pairwise distinct [Bartholdi – Grigorchuk, 2000].

Weakly maximal subgroups of branch groups

Question (Grigorchuk,2005)

Describe all weakly maximal subgroups of $\mathfrak{G}$.

- (Pervova, 2011) Concrete example of a weakly maximal subgroup W_P of $\mathfrak{G}$ which is not parabolic.
- (Bou-Rabee – L. – Nagnibeda, 2016) If G is branch and contains a finite subgroup F that fixes no rays, then it contains uncountably many non parabolic weakly maximal subgroups (non-constructive proof).
- (L., 2019) Complete description of the weakly maximal subgroups of $\mathfrak{G}$ and of torsion $\mathtt{GGS}$ groups.

The main theorem

Theorem (L., 2019)

Let $\mathcal{G}$ be either the first Grigorchuk group, or a torsion GGS *group. Weakly maximal subgroups of $\mathcal{G}$ belong to one of the following two classes: generalized parabolic subgroups and weakly maximal subgroups with a block structure. These two classes admit many characterization, as shown in the following table.*

generalized parabolic	*with block structure*
finite type	*not finite type*
$\forall v : \mathrm{Rist}_W(v)$ *is infinite*	$\exists v : \mathrm{Rist}_W(v) = \{1\}$
$W \curvearrowright \partial T$ *has infinitely many orbits-closure*	$W \curvearrowright \partial T$ *has finitely many orbits-closure*
$\forall n \exists v \in \mathcal{L}_n : [\pi_v(\mathcal{G}) : \pi_v(W)]$ *is infinite*	$\exists n \forall v \in \mathcal{L}_n : [\pi_v(\mathcal{G}) : \pi_v(W)]$ *is finite*

Remarks on the main theorem

- Let G be a finitely generated branch group. Then every finite index subgroup H of G satisfies
  1. H is of finite type;
  2. $\forall v$: $\mathrm{Rist}_H(v)$ is infinite;
  3. $H \curvearrowright \partial T$ has finitely many orbits-closure;
  4. $\exists n \forall v \in \mathcal{L}_n : [\pi_v(\mathfrak{G}) : \pi_v(H)]$ is finite.
- If $W \leq \mathcal{G}$ is weakly maximal, then either it satisfies both Property 1 and 2 or it satisfies both Properties 3 and 4.

Two types of weakly maximal subgroups

Generalized parabolic subgroups

Definition

A generalized parabolic subgroup of $G \leq \mathrm{Aut}(T)$ is a setwise stabilizer $\mathrm{SStab}_G(C)$ where

- $C \subseteq \partial T$ is closed;
- C has empty interior (i.e. is nowhere dense);
- the action of $\mathrm{SStab}_G(C)$ on C is minimal.

Example

- $C = \{\xi\}$ for $\xi \in \partial T$;
- $C = F.\{\xi\}$ where F is a finite subgroup of G.

Properties

Lemma

Let $G \leq \mathrm{Aut}(T)$ be a branch group and let C be a closed subset of ∂T. Then $\mathrm{SStab}_G(C)$ is weakly maximal if and only if it is generalized parabolic.

Lemma

Let G be branch. Then generalized parabolic subgroups are infinite and pairwise distinct ($\mathrm{SStab}_G(C_1) \neq \mathrm{SStab}_G(C_2)$ if $C_1 \neq C_2$).

Corollary

Any branch group with an element of finite order contains a continuum of generalized parabolic subgroups that are not parabolic (they are all weakly maximal).

Proof.

Take $g \in G$ of finite order and look at the $\langle g \rangle$-orbits on ∂T. □

Properties (2)

Let $G \leq \mathrm{Aut}(T)$ be a branch group and $W \leq G$ be a generalized parabolic subgroup. Then

- all the $\mathrm{Rist}_W(v)$ are infinite;
- the action $W \curvearrowright \partial T$ has infinitely many orbits-closure.

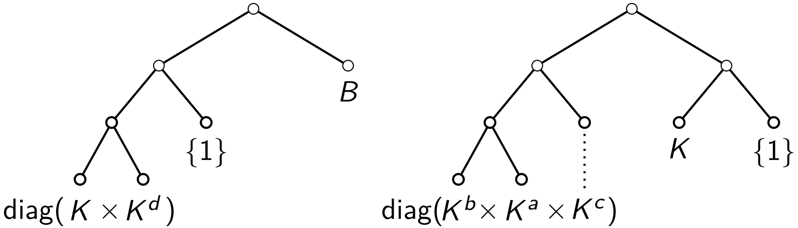
Another example

Let $G \leq \mathrm{Aut}(T)$ be a regular branch group over K. If G is torsion, then there exists a continuum of generalized parabolic subgroups of G of the form $\mathrm{SStab}_G(C)$ where C is a nowhere dense Cantor subset of ∂T.

Block subgroups: examples

Recall that the first Grigorchuk group $\mathfrak{G} = \langle a, b, c, d \rangle$ is regularly branch over K and we have $K <_2 B = \langle b \rangle^{\mathfrak{G}} <_8 \mathfrak{G}$.

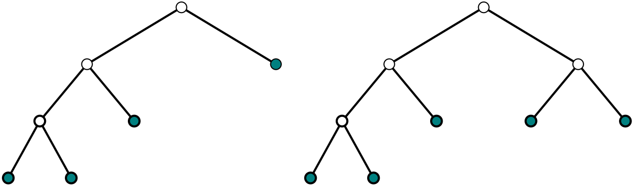
Example



Block subgroups

Let $G \leq \mathrm{Aut}(T)$ be self-replicating branch group. A block subgroup of G is given by

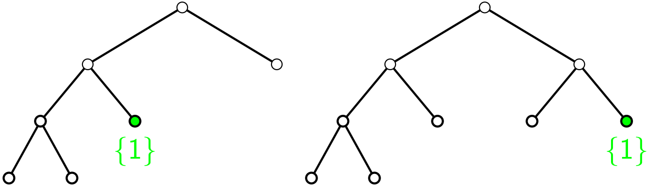
- a transversal U of T (every $\xi \in \partial T$ passes through exactly 1 vertex of U);



Block subgroups

Let $G \leq \mathrm{Aut}(T)$ be self-replicating branch group. A block subgroup of G is given by

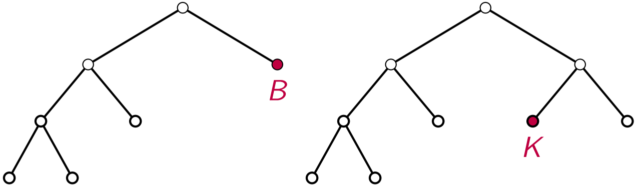
- a transversal U of T;
- zero or more trivial blocks;



Block subgroups

Let $G \leq \mathrm{Aut}(T)$ be self-replicating branch group. A block subgroup of G is given by

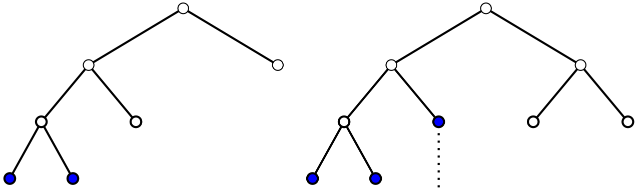
- a transversal U of T;
- zero or more trivial block;
- zero or more full block: finite index subgroup of G (more precisely of $\pi_v(\mathrm{Rist}_G(v))$;



Block subgroups

Let $G \leq \mathrm{Aut}(T)$ be self-replicating branch group. A block subgroup of G is given by

- a transversal U of T;
- zero or more trivial block;
- zero or more full block;
- zero or more diagonal blocks;

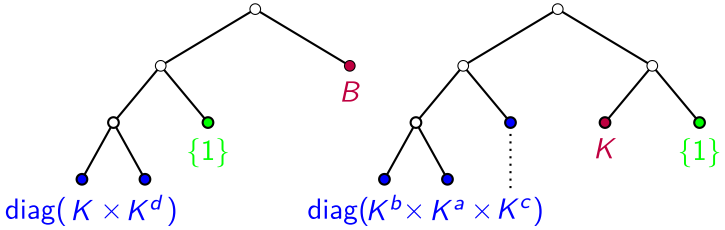


$\mathrm{diag}(K \times K^{d})$ $\mathrm{diag}(K^{b} \times K^{a} \times K^{c})$

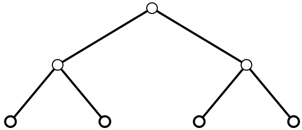
Block subgroups

Let $G \leq \mathrm{Aut}(T)$ be self-replicating branch group. A block subgroup of G is given by

- a transversal U of T;
- zero or more trivial block;
- zero or more full block;
- zero or more diagonal blocks;
- such that U is the union of vertices of trivial, full and diagonal blocks.



Diagonal subgroup: an example



$$\mathrm{diag}(K^a \times K^d) = \{(k^a, k^d, 1, 1) \mid k \in K\}$$

Diagonal subgroups: definition

Let $G \leq \mathrm{Aut}(T)$ be a self-replicating branch group. A diagonal subgroup D of G is given by $(\{v_i\}_{i=1}^n, H, (H_i)_{i=1}^n, \{\psi_i\}_{i=1}^n)$ where $n \geq 2$ and

- the v_i are pairwise orthogonal vertices (if $i \neq j$, we have both $v_i \not\leq v_j$ and $v_j \not\leq v_i$);
- H is a finite index subgroup of G;
- $H_i \leq G \leq \mathrm{Aut}(T_{v_i})$;
- $\psi_i \colon H \to H_i$ is an isomorphism.

$$\begin{aligned} D &= \mathrm{diag}(\psi_1(H_1) \times \cdots \times \psi_n(H_n)) \\ &= \{g \in \prod_{i=1}^n \mathrm{Rist}(v_i) \mid \exists h \in H : \forall i\, \pi_{v_i}(g) = \psi_i(h)\} \end{aligned}$$

Block subgroups: properties

Let B be a block subgroup of a finitely generated, self-replicating branch group $G \leq \mathrm{Aut}(T)$. Then

- B is finitely generated;
- If B has no trivial blocks and at least one diagonal block, then it is of infinite index and every weakly maximal subgroup W containing B is not generalized parabolic;
- In particular, there exists infinitely many weakly maximal subgroups of G that are not generalized parabolic.

Block structure

Definition

Let G be a self-replicating branch group. A subgroup H of G is said to have a block structure if it contains a block subgroup B such that $[H : B] < \infty$.

Lemma

If W is a weakly maximal subgroup with block structure, then

- *there exists v such that* $\mathrm{Rist}_W(v) = \{1\}$;
- *there exists n such that for every vertex v of level n we have* $[\pi_v(G) : \pi_v(W)] < \infty$.

Technical tools

The non-rigidity tree

An "inverse" operation to $\mathrm{SStab}_G(\cdot)$

Let $G \leq \mathrm{Aut}(T)$ be a branch group and $H \leq G$.

Definition

The non-rigidity tree $\mathrm{NR}(H)$ of H is the subgraph of T generated by the vertices v such that $\mathrm{Rist}_H(v)$ has infinite index in $\mathrm{Rist}_G(v)$.

- $\mathrm{NR}(H)$ is a tree containing the root of T (unless H is of finite index in which case $\mathrm{NR}(H) = \emptyset$);
- If W is weakly maximal, then $\mathrm{NR}(W)$ has no leaf;
- If $W = \mathrm{SStab}_G(C)$ is generalized parabolic, then $C = \partial\,\mathrm{NR}\big(\mathrm{SStab}_G(C)\big)$ and $W = \mathrm{SStab}_G\big(\mathrm{NR}(W)\big)$;

Tree-equivalence

Definition

Two weakly maximal subgroups W_1 and W_2 of $G \leq \mathrm{Aut}(T)$ are tree-equivalent if there exists $\varphi \in \mathrm{Aut}(T)$ such that $\mathrm{NR}(W_2) = \varphi(\mathrm{NR}(W_1))$.

- The class of parabolic subgroups is a tree-equivalence class;
- The class of generalized parabolic subgroups is a union of tree-equivalence classes.

Many tree-equivalence classes

Proposition

Let G be a torsion branch group. Then

- *It contains a continuum of tree-equivalence classes of generalized parabolic subgroups, each classes containing infinitely many subgroups;*
- *It contains infinitely many tree-equivalence classes of generalized parabolic subgroups, each classes containing a continuum of subgroups.*

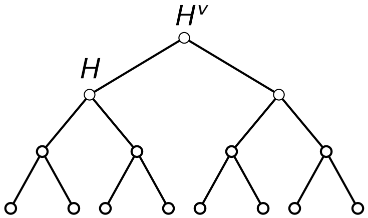
Induction of weakly maximal subgroups

An "inverse" operation to $\pi_v(\cdot)$

Definition

Let $G\,\mathrm{Aut}(T)$ be a self-replicating group, $v \in T$ and $H \leq G$. The subgroup of G induced by H is

$$H^v := \{g \in \mathrm{Stab}_G(v) \mid \pi_v(g) \in H\}$$

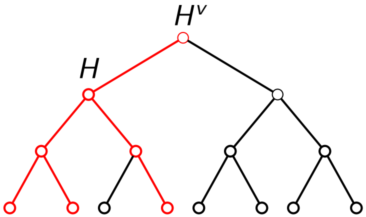


Properties

Proposition (L., 2019)

Let $G \leq \mathrm{Aut}(T)$ be a finitely generated, self-replicating branch group, v a vertex of T and $H \leq G$. Then

- $\mathrm{NR}(H^v) = [\emptyset, v] \cup \mathrm{NR}(H)$;



- $[G : H^v]$ *is finite if and only if* $[G : H]$ *is finite;*
- H^v *is weakly maximal if and only if* H *is weakly maximal;*
- H^v *is finitely generated if and only if* H *is finitely generated;*
- $\pi_v(H^v) = H$ *and* $(\pi_v(H))^v \geq H$. *Moreover, if* H *is weakly maximal and* $\mathcal{L}_n \cap \mathrm{NR}(H) = \{v\}$, *then* $(\pi_v(H))^v = H$. *(i.e. you can cut the beginning of the trunk of* $\mathrm{NR}(H)$*, or add some*

The induction subgroup property

Theorem (Grigorchuk-L.-Nagnibeda, 2020)

Let G be a finitely generated self-replicating branch group. Then G has the induction subgroup property if and only if finitely generated subgroups of G coincide with subgroups with a block structure.

Theorem

The following groups have the subgroup induction property:

- *the first Grigorchuk group (Grigorchuk – Wilson, 2003);*
- *the Gupta-Sidki 3-group (Garrido, 2016);*
- *torsion GGS groups (Francoeur – L., 2020^+).*

Proposition (Francoeur – L., 2020^+)

If G has the subgroup induction property, then it is torsion and just infinite.

A technical result

Theorem (L., 2019)

Let $G \leq \mathrm{Aut}(T)$ be a f. g. branch group and $W \leq G$ a weakly maximal subgroup. For the following properties that W may have,

(a) W has a block structure

(b) $\exists n \forall v \in \mathcal{L}_n : [\pi_v(G) : \pi_v(W)] < \infty$

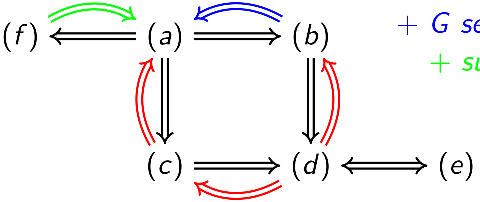
(c) $\exists v : \mathrm{Rist}_W(v) = \{1\}$

(d) $W \curvearrowright \partial T$ has $< \infty$ orbits closure

(e) W not generalized parabolic

(f) $\mathrm{rank}(W) < \infty$

we have the implications



+ G self-replicating and just infinite

+ subgroup induction property

$G = \mathfrak{G}$ or torsion GGS

Open questions/problems

Open questions and problems

- This is **not** a classification!
- These methods can only produce closed (in the profinite topology) subgroups, and hence cannot detect maximal subgroups of infinite index.
- Conjecture: The main theorem holds in any finitely generated, self-replicating branch group with the subgroup induction property.

THANK YOU!