Cayley graphs with few automorphisms

Paul-Henry Leemann (Neuchâtel)
Joint work with Mikael de la Salle (Lyon)

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What is it all about

A subject at the intersection of

- Geometric Group Theory,
- Combinatorics and Graphs,
- Probability and Random Walks.

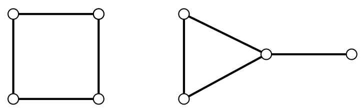
Generally known under the name of *Graphical Rigid Representation*.

Representation of groups

- A (classical) representation of a group G is an homomorphism $G \to \mathrm{GL}(V)$, where V is a vector space over $\mathbf{C}$,
- More generally, one can look at $G \to \mathrm{Aut}(X)$, where X is a *geometric space with good properties*.
- In our case, we will look at $G \cong \mathrm{Aut}(X)$ with X a connected graph.

Graphs

- A graph X is made of a set V of vertices and of a set E of edges.



- A graph X is connected if for every pair of vertices (v, w) there is a path from v to w.
- A graph X is locally finite if any vertex has only finitely many adjacent edges.

A first question

Question

What are the finitely generated groups G such that there exists a connected locally finite graph X with $G = \mathrm{Aut}(X)$.

- All finitely generated groups [Groot (1959) and Sabidussi (1960)].
- What happens if we restrict ourself to graphs with more structure (for example: of regular degree)?

Regular graphs

Definition

The action of $\mathrm{Aut}(X)$ on X is regular if it is free and transitive on the vertices. That is, for every par of vertices (v, w) there exists a unique automorphism of X sending v to w.

Main question

Question

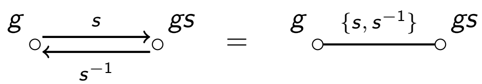
What are the finitely generated groups G such that there exists a connected locally finite graph X with $G = \mathrm{Aut}(X)$ acting ***regularly*** *on X.*

- In this case, X is a Cayley graph of G [Sabidussi, 1958].
- Solved for finite groups in the 70' [Imrich, Watkins, Nowitz, Hetzel, Godsil...].
- Solved for free products of finitely generated groups [Watkins, 1976].
- Solved [L. - de la Salle] in 2021-2022 for finitely generated infinite groups.

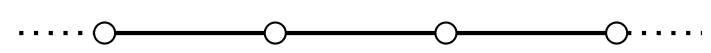
Cayley graphs

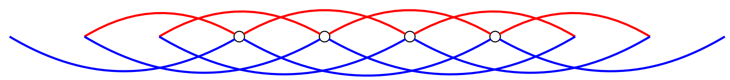
Definition

Let G be a group and $S = S^{-1}$ be a generating set. The corresponding Cayley graph is the graph with vertices set $V = G$ and with, for every $g \in G$ and $s \in S$, an arc, labeled by s, from g to gs.



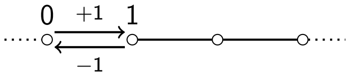
Example

- $\mathrm{Cayl}(\mathbf{Z}, \{\pm 1\}) =$ 
- $\mathrm{Cayl}(\mathbf{Z}, \{\pm 2, \pm 3\}) =$



Cayley graphs

- Each edge consists of a pair of arcs.
- Each arc has a label ($s \in S$).
- The colour of an edge is the pair of its labels ($\{s, s^{-1}\} \subset S$).

$\mathrm{Cayl}(\mathbf{Z}, \{\pm 1\}) =$ 

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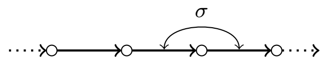
- $G \curvearrowright \mathrm{Cayl}(G, S)$ by left multiplication.
- We have

$$\begin{aligned} G &= \mathrm{Aut}_{\mathrm{lab}}(\mathrm{Cayl}(G,S)) \\ &\leq \mathrm{Aut}_{\mathrm{col}}(\mathrm{Cayl}(G,S)) \\ &\leq \mathrm{Aut}(\mathrm{Cayl}(G,S)) = G \cdot \mathrm{Stab}_{\mathrm{Aut}(\mathrm{Cayl}(G,S))}(1). \end{aligned}$$

Example for $\mathbf{Z}$

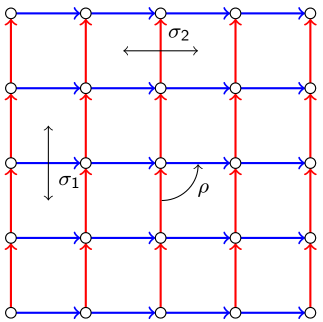


$\mathrm{Aut}_{\mathrm{lab}}(\mathrm{Cayl}(\mathbf{Z}, S)) = \mathbf{Z}$

$\mathrm{Aut}_{\mathrm{col}}(\mathrm{Cayl}(\mathbf{Z}, S)) = \mathbf{Z} \rtimes \{1, \sigma\} = D_\infty$

$\mathrm{Aut}(\mathrm{Cayl}(\mathbf{Z}, S)) = D_\infty$

Example for $\mathbf{Z}^2$



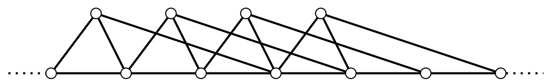
$\mathrm{Aut}_{\mathrm{lab}}(\mathrm{Cayl}(\mathbf{Z}^2, S)) = \mathbf{Z}^2$

$$\begin{aligned} \mathrm{Aut}_{\mathrm{cou}}(\mathrm{Cayl}(\mathbf{Z}^2, S)) &= \langle \mathbf{Z}^2, \sigma_1, \sigma_2 \rangle \\ &= \mathbf{Z}^2 \rtimes (\mathbf{Z}/2\mathbf{Z})^2 \end{aligned}$$

$$\begin{aligned} \mathrm{Aut}(\mathrm{Cayl}(\mathbf{Z}^2, S)) &= \langle \mathbf{Z}^2, \sigma_1, \sigma_2, \rho \rangle \\ &= \mathbf{Z}^2 \rtimes D_{2\cdot 4} \end{aligned}$$

A graph with $\mathrm{Aut}(X) = \mathbf{Z}$

We begin with $X = \mathrm{Cayl}(G, S)$ to which we add decorations in order to *fix the orientation.*



Exercice: find X with $\mathrm{Aut}(X) = \mathbf{Z}^2$.

Main question (bis repetita)

Question

What are the finitely generated groups G such that there exists a finite, symmetric, generating set S with $G = \mathrm{Aut}(\mathrm{Cayl}(G, S))$?

When $G = \mathrm{Aut}(\mathrm{Cayl}(G, S))$, we say that $\mathrm{Cayl}(G, S)$ is a graphical regular representation (GRR), und that G is rigid if there exists such an S.

Non-rigid groups

Fact

If G is abelian and is not isomorphic to $(\mathbf{Z}/2\mathbf{Z})^n$, then it is not rigid. Indeed, the map $g \mapsto g^{-1}$ is an automorphism of $\mathrm{Cayl}(G, S)$ for every S.

G is generalized dicyclic if it is not abelian and $G = A \sqcup xA$ with A an abelian subgroup, x of order 4 and $xax^{-1} = a^{-1}$ for every $a \in A$. Example: $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$.

Fact

If G is a generalized dicyclic group, then it is not rigid. The map $a \mapsto a$, $xa \mapsto a^{-1}x^{-1}$ is an automorphism of $\mathrm{Cayl}(G, S)$ for every S.

Fact

There exists 13 exceptional groups of order at most 32 that are not rigid (nor in one of the above two infinite families).

Rigid groups

Theorem (Imrich, Watkins, Nowitz, Hetzel, Godsil..., 1969-1981)

Let G be a finite group. If G is neither generalized dicyclic, nor abelian (not isomorphic to $(\mathbf{Z}/2\mathbf{Z})^n$) nor one of the 13 exceptional groups, then it is rigid.

- No unified construction, but a lot of distinct cases.
- Use strongly the fact that G is finite (Feit-Thompson, ...).

Theorem (Watkins,1976)

If $G = G_1 * \cdots * G_n$ is a free product of finitely generated groups, then it is rigid.

Asymptotic

Theorem (Babai-Godsil, 1982)

If G is nilpotent, non-abelian, finite of even order, then asymptotically almost all Cayley graphs of G are GRR.

Main Result

Theorem (L. - de la Salle, 2021-2022)

Let G be a finitely generated infinite group. If G is neither generalized dicyclic nor abelian, then it is rigid.
Moreover, for every finite generating set S, there exists $S \subset T$ such that $\mathrm{Cayl}(G, T)$ is a GRR (with $|T| \leq f(|S|)$ for some explicit f).

- A unique common structure for the proof, with only two cases;
- The proof also works for finite groups with an element of *big* order (depending on $\mathrm{rank}(G)$). In particular, we reobtain that for every n there exists only finitely many exceptional groups of rank n (use Zelmanov solution to the restricted Burnside problem).
- Can be thoughts as a (very) weak form of asymptotic result.

Main idea

- Remind:

$$G = \mathrm{Aut}_{\mathrm{lab}}(\mathrm{Cayl}(G,S)) \leq \mathrm{Aut}_{\mathrm{col}}(\mathrm{Cayl}(G,S)) \leq \mathrm{Aut}(\mathrm{Cayl}(G,S)).$$

- Starting with S, we will construct $S \subset T \subset U$ and check separately that both of the above inequalities are in fact equalities.

Structure of the proof

Proposition 1

Let G be a group that is neither generalized dicyclic nor abelian (not isomorphic to $(\mathbf{Z}/2\mathbf{Z})^n$). Then for every generating set S, there exists $S \subset T$ (finite if S is finite) such that $\mathrm{Aut}_{\mathrm{col}}(\mathrm{Cayl}(G, T))$ preserves the S-labels.

Proposition 2

Let G be a finitely generated infinite group. Then for every finite generating set T, there exists $T \subset U$ finite such that $\mathrm{Aut}(\mathrm{Cayl}(G, U))$ preserves the T-colours.

Proposition 3

Let $S \subset T \subset U$ be as above. Then $\mathrm{Cayl}(G, U)$ is a GRR for G.

Proof of Proposition 3

Let φ be an element of $\mathrm{Aut}(\mathrm{Cayl}(G, U))$. Up to composing by an element of $\mathrm{Aut}_{\mathrm{lab}}(\mathrm{Cayl}(G, U))$, we can suppose that φ fixes the root of $\mathrm{Cayl}(G, U)$.
The restriction of φ belongs to $\mathrm{Aut}_{\mathrm{col}}(\mathrm{Cayl}(G, T))$ by Proposition 2 and thus also to $\mathrm{Aut}_{\mathrm{lab}}(\mathrm{Cayl}(G, S))$ by Proposition 1.
The restriction of φ fixes the root and preserves the S-labels. Since S is generating, φ fixes all the vertices and is hence the identity. We conclude that φ is in $\mathrm{Aut}_{\mathrm{lab}}(\mathrm{Cayl}(G, U))$.

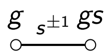
Sketch of a proof of Proposition 1

- Let G be a group, $S = S^{-1}$ be a generating set and $T = (S \cup S^2 \cup S^3) \setminus \{1\}$.
- We look at the subgroup H of $\mathrm{Aut}_{\mathrm{col}}(\mathrm{Cayl}(G, T))$ consisting of automorphisms fixing the vertex 1_G.
- These are the bijections $\varphi \colon G \to G$ satisfying

$$\varphi(1) = 1 \text{ et } \forall g \in G, \forall t \in T, \varphi(gt) \in \varphi(g)\{t, t^{-1}\}$$

- We show that if H does not fixe pointwise S, then G is abelian or generalized dicyclic. The proof is mainly combinatorics and the quaternion group Q_8 plays a central role.

Proof of Proposition 2: Triangles

- We will use a geometric invariant to distinguish between an edge coloured by $\{s^{\pm 1}\}$ and an edge coloured by $\{t^{\pm 1}\}$: the number of triangles to which they belong.
- For $s \in S$, wed denote by $\mathrm{Tr}(s, S)$ the number of triangles of $\mathrm{Cayl}(G, S)$ containing the edge 

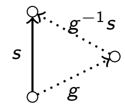
(does not depend on g).
- We always have $\mathrm{Tr}(s, S) = \mathrm{Tr}(s^{-1}, S)$.

Proof of Proposition 2

- Given a finite S, we will construct $S \subset T$ finite and such that:
  - For every $t \in T \setminus S$ we have $\mathrm{Tr}(t, T) \leq 6$,
  - For every $s \in S$ we have $\mathrm{Tr}(s, T) \geq 7$,
  - For every $s, s' \in S$ we have $\mathrm{Tr}(s, T) = \mathrm{Tr}(s', T)$ if and only if $s' = s$ or $s' = s^{-1}$.
- To do that, we will show a technical lemma saying that we can augment the number of triangles to which belongs $s_0 \in S$ without augmenting the number of triangles to which belong elements of $S \setminus \{s_0, s_0^{-1}\}$.
- By applying this lemma several times we are done.

Technical Lemma

Let s be an element of S.

- For every $g \in G$, we look at $S_g = S \cup \{g, g^{-1}, g^{-1}s, s^{-1}g\}$.
- 

- We want $g \in G$ such that:
  - We augment the number of triangles to which belongs s $(\mathrm{Tr}(s, S_g) > \mathrm{Tr}(s, S_g))$;
  - $\mathrm{Tr}(g, S_g) \leq 6$ and $\mathrm{Tr}(g^{-1}s, S_g) \leq 6$;
  - We do not augment the number of triangles to which belong $t \in S \setminus \{s, s^{-1}\}$.
- This gives us a list of conditions: $g \notin S$, $s^{-1}g \notin S$, ...

An algebraic criterion

At the end we obtain the following criterion:
There exists a finite $F \subset G$ such that if $g, s^{-1}g \notin F$ and $g^2, (s^{-1}g)^2 \notin F$, then S_g works.
Let $\mathrm{sq}\colon G \to G,\ g \mapsto g^2$ be the square map. Then $\mathrm{sq}^{-1}(F)$ is the subset of elements $g \in G$ such that $g^2 \in F$.

Dichotomy

For the rest of the proof, we will treat separately two cases:

- G has an element of infinite order (or of order *sufficiently big*),
- G is not virtually abelian.

Remind: G is virtually abelian if it contains an abelian subgroup H of finite index. For example, every finite group is virtually abelian.
Moreover, if G is virtually abelian and finitely generated, then either it is finite or it has an element of infinite order.

G has an element of infinite order

Let $g_0 \in G$ be of infinite order.

- We restrict ourself to g in $\langle g_0 \rangle \cong \mathbf{Z} \leq G$.
- In $\mathbf{Z}$, every element has at most one square root.
- Therefore, there exists infinitely many g in $\langle g_0 \rangle$ such that both $g, s^{-1}g \notin F$ and $g^2 \notin F$.
- With a little more work, we obtain the desired result, except the fact the when augmenting triangles for s, we might also augment the triangles for s^2.
- If we are careful enough (first apply the lemma to s and then to s^2), this is not a problem.

G is not virtually abelian

For an arbitrary G and $F \subset G$ finite, it may happen that $\mathrm{sq}^{-1}(F)$ is infinite; it is therefore not possible to use the above strategy without modification.
But, we can show

Proposition 4

Let G be a finitely generated non virtually abelian group. For every $s \in S$ and every finite $F \subseteq G$, the set $G \setminus (\mathrm{sq}^{-1}(F) \cup s\,\mathrm{sq}^{-1}(F))$ is infinite.

Corollary

Let G be a finitely generated non virtually abelian group. For every $s \in S$ and every finite $F \subseteq G$, there exists $g \in G$ such that

$$g, s^{-1}g \notin F \quad \text{and} \quad g^2, (s^{-1}g)^2 \notin F.$$

Proof of Proposition 4

In order to prove Proposition 4, we will use

- If G is finitely generated and every element has order at most 2, then G is finite,
- A lemma, due to Dicman, about normal subgroups,
- Random walks on groups, including a result due to Tointon.

A lemma of Dicman

Lemma (Dicman)

Let G be a group and $F \subset G$ be a finite subset. If every element of F has finite order and F is invariant by conjugation, then the normal subgroup $\langle F \rangle^G$ is finite

An application of Dicman's lemma

Corollary

Let G be a finitely generated group. Then G is finite if and only if $\mathrm{sq}(G)$ *is finite.*

Proof.

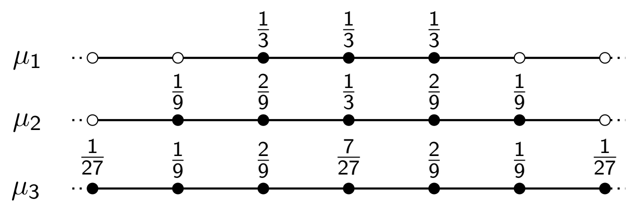
Let $F = \mathrm{sq}(G)$. This subset is closed under conjugation. If F is finite, then it contains only elements of finite order. The group $G/\langle F \rangle^G$ is finitely generated and all of its elements have order at most 2, it is hence finite. But by Dicman $\langle F \rangle^G$ is also finite, hence G itself is finite. □

Random walks on groups

Let G be a finitely generated group and $S = S^{-1}$ be a finite generating set containing 1.
Let μ be the uniform probability of choosing an element of S and $\mu_n = \mu^{*n}$ the corresponding random walk.

Example

$G = \mathbf{Z}$ and $S = \{-1, 0, 1\}$



A theorem of Tointon

Theorem (Tointon, 2020)

Let G be a finitely generated group, $S = S^{-1}$ be a finite generating set containing 1 and μ be the uniform probability on S. Let g_n and h_n be two independant realizations of μ_n. If G is not virtually abelian, then

$$\lim_{n\to\infty} \mathbf{P}(g_n \text{ and } h_n \text{ commute}) = 0.$$

Corollary (L.-dlS.)

Same hypothesis. If G is not virtually abelian, then

$$\liminf_{n\to\infty} \mathbf{P}(g_n^2 = 1) \leq \frac{\sqrt{5}-1}{2}.$$

With more works, we prove Proposition 4.

Variations on a theme 1

One can ask the question of what happens for directed graphs. For a, non necessarily symmetric, generating set $S \subset G$, we define $\overrightarrow{\mathrm{Cayl}}(G, S)$ in an analogous way as $\mathrm{Cayl}(G, S)$.

Question

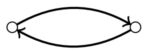
What are the finitely generated groups G such that there exists a finite and generating S with $G = \mathrm{Aut}(\overrightarrow{\mathrm{Cayl}}(G, S))$? (DRR)

- Easier than finding GRR,
- All finite groups, with 5 exceptions (Babai, 1980),
- All infinite groups, but with S infinite (Babai, 1980),
- All finitely generated infinite groups (L.-dlS.).

Variations on a theme 2

Question (Babai, 1980)

What are the finitely generated groups G such that there exists a finite and generating S with $S \cap S^{-1} = \emptyset$ and $G = \mathrm{Aut}(\overrightarrow{\mathrm{Cayl}}(G, S))$? (ORR)

- The condition $S \cap S^{-1} = \emptyset$ says that each edge can be followed in only one direction, i.e. we don't have  .
- If G is generalized dihedral ($G = A \rtimes \mathbf{Z}/2\mathbf{Z}$ with A abelian), this is not possible. Indeed, in this case every generating set of G contains an element of order 2.
- All finite groups that are not generalized dihedral, with 11 exceptions (Morris-Spiga, 2018).
- All finitely generated infinite groups that are not generalized dihedral (L.-dlS.).

Other consequences 1

Corollary

Every finitely generated group admits a locally finite Cayley graph with a countable group of automorphisms (equivalently such that the vertex stabilizers are finite).

This answer a conjecture of dlS. and Tessera (2019).

Other consequences 1

A graph X is LG-rigid (local to global) if there exists an integer r such that if Y is a graph with the same balls of radius r as X, then X covers Y.

Corollary

Every finitely presented group admits a locally finite LG-rigid Cayley graph.

A group which is not finitely presented does not have LG-rigid Cayley graphs (dlS-Tessera, 2019). In particular, the above corollary gives a new characterization of finitely presented groups.