De Bruijn Graphs, Lamplighter Groups and Spectral Computations

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June 12, 2015
Joint AMS-EMS-SPM Meeting, Porto

Convergence of finite graphs

Let $\{\Gamma_n\}_n$ be a sequence of finite transitive graphs of bounded degree. We say that a graph Γ is the limit of $\{\Gamma_n\}_n$ if for every r, there exists N such that for all $n > N$, the ball of radius r in Γ_n is isomorphic to the ball of radius r in Γ.
If $\{\Gamma_n\}_n$ are not transitive, one has to consider convergence of rooted graphs. The limit depends on the choice of the roots $v_n \in \Gamma_n$!

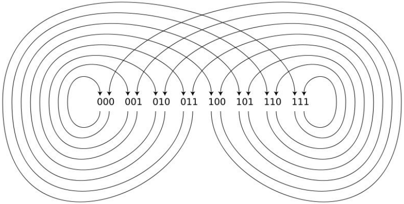
Definition (Benjamini-Schramm)

Let $\{\Gamma_n\}_n$ be an arbitrary sequence of finite graphs of bounded degree. One can consider them as rooted graphs by choosing a root in each Γ_n uniformly at random. This defines a sequence of probability measures on the space of (isomorphism classes of) rooted graphs, and one can consider its weak limit and call it the Benjamini-Schramm limit of the sequence $\{\Gamma_n\}_n$.

The Benjamini-Schramm limit is a probability distribution on the space of rooted graphs supported by the limits of the sequence of graphs $\{\Gamma_n\}_n$ for all possible choices of roots $v_n \in \Gamma_n$.

De Bruijn Graphs

An n-dimensional De Bruijn graph on k symbols, $B_{k,n}$, is a directed graph representing overlaps between sequences of symbols. The vertex set is $\{0, 1, \ldots, k-1\}^n$, and for all $0 \leq i \leq k-1$, there is an oriented edge from $(a_1, \ldots, a_n)$ to $(a_2, \ldots, a_n, i)$.



The De Bruijn graph $B_{2,3}$.

De Bruijn graphs $B_{k,n}$ are discrete models of the Bernoulli map $x \mapsto kx \pmod 1$ and therefore are of interest in the theory of dynamical systems. They also have applications in informatics (for peer-to-peer file sharing and parallel computing) and bioinformatics (genome assembly algorithms).

Limit of De Bruijn Graphs

Question

What is the Benjamini-Schramm limit of the sequence $B_{k,n}$ as n goes to infinity?

Observe that the De Bruijn graphs are not vertex transitive.

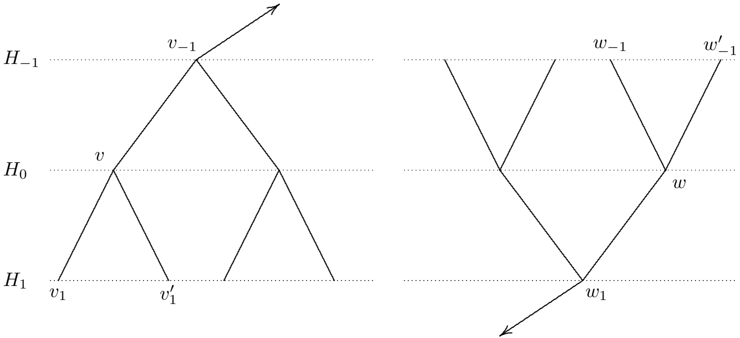
Theorem (Grigorchuk - Leemann - Nagnibeda)

The Benjamini-Schramm limit of the sequence $\{B_{k,n}\}_n$ is the Diestel-Lieder graph $\mathrm{DL}(k,k) \cong \mathrm{Cay}(L_k, X)$.

Here $L_k = \mathbf{Z}/k\mathbf{Z} \wr \mathbf{Z} = (\bigoplus_{\mathbf{Z}} \mathbf{Z}/k\mathbf{Z}) \rtimes \mathbf{Z}$ is the Lamplighter group on $\mathbf{Z} = \langle t \rangle$ with lamp group $\mathbf{Z}/k\mathbf{Z} = \langle a \rangle$ and generating system $X = \{t, at, \ldots, a^{k-1}t\}$.

In fact, $\mathbf{Z}/k\mathbf{Z}$ could be replaced by any group with k elements.

GLN also studied a two-parameter generalization of the family $\{B_{k,n}\}$, the so-called spider-web networks, introduced in theory of networks and studied by physicists for their properties in percolation.



The graph $\mathrm{DL}(k,k)$ is the horocyclic product of two $k+1$ regular trees. Vertices correspond to couples (v,w) of same height.

Here, the vertex (v,w) has 4 neighbors: (v_1,w_1), (v'_1,w_1), (v_{-1},w_{-1}) and (v_{-1},w'_{-1}).

Corollary

The De Bruijn graphs $B_{k,n}$ provide a sofic approximation of $\mathrm{DL}(k,k) \cong \mathrm{Cay}(L_k, X)$.

The proof of the theorem goes via the following basic fact:

The group L_k acts faithfully on a rooted k-regular tree T_k.

This action is transitive on each level and gives rise to Schreier orbital graphs Γ_n (with respect to the generators X).

Grigorchuk and Żuk (2001, for $k = 2$) and Kambites, Silva and Steinberg (2006, general case) showed that the action is given by an automaton and that te extension of this action to the boundary ∂T_k is essentially free. This implies that the Benjamini-Schramm limit of Γ_n is $\mathrm{Cay}(L_k, X)$.

We proved that the De Bruijn graph $B_{k,n}$ is isomorphic to Γ_n as unlabeled graph.

There is now also an independent combintorial proof by P.-H. Leemann that $B_{k,n} \to_{n\to\infty} \mathrm{DL}(k,k)$ that hopefully can be generalized to other families of Rauzy graphs.

Spectral theory

Grigorchuk-Żuk used the action of L_k on the tree to compute the spectral measure of the Laplacian on $\mathrm{Cay}(L_2, X)$:

$$\mu_{\mathrm{Cay}(L_k,X)} = (k-1)^2 \sum_{q\geq 2} \frac{1}{k^q - 1} \Bigg(\sum_{\substack{1\leq p<q \\ (p,q)=1}} \delta_{2k(1-\cos(\frac{p}{q}\pi))} \Bigg).$$

This was the first example of an infinite transitive graph with discrete spectral measure.
The convergence of the De Bruijn graphs implies convergence of the spectral measure in the following sense:

$$\frac{1}{k^n} \sum_{i=1}^{k^n} \delta_{\lambda_i} \xrightarrow{n\to\infty} \mu_{\mathrm{Cay}(L_k,X)}$$

where the λ_i are the eigenvalues of the Laplacian on $B_{k,n}$ and the convergence is the weak convergence of measures.

Complexity

A natural invariant that one can associate to a finite graph Γ is the number $t(\Gamma)$ of covering trees. This number is called the complexity of the graph. For $\{\Gamma_n\}_n$ a sequence of finite graphs, one could try to define the asymptotic complexity as

$$t_{as} = \lim_n \frac{\log(t(\Gamma_n))}{|V(\Gamma_n)|}.$$

In 2003, R. Lyons showed that the limit always exists and can be directly computed on the Benjamini-Schramm limit of the sequence $\{\Gamma_n\}$. If this limit is a transitive graph Γ of degree d, then

$$t_{as} = \log(d) - \sum_{j\geq 1} \frac{1}{j} p_j(o;\Gamma),$$

where $p_k(o;\Gamma)$ denotes the probability that the simple random walk started at o is back at o after k steps.

Complexity and Laplacian

The Kirchhoff's matrix tree theorem says that if Γ is a finite connected graph, then $t(\Gamma) \cdot |V(\Gamma)|$ is equal to the product of the non-zero eigenvalues of the Laplacian of Γ.
In other words, if we define the spectral zeta function of Γ as

$$\zeta_\Gamma(s) = \sum_{\lambda \in \mathrm{Spec}\,\Delta \setminus \{0\}} \lambda^{-s},$$

Kirchhoff's formula says

$$t(\Gamma) \cdot |V(\Gamma)| = e^{-\zeta'_\Gamma(0)} = \tilde{\mathrm{Det}}(\Delta_\Gamma).$$

By analogy, one can define for Γ infinite, the spectral zeta function of Γ as

$$\zeta_\Gamma(s) = \int_{\mathrm{Spec}(\Delta)} \lambda^{-s} \, \mathrm{d}\mu_\Gamma(\lambda).$$

In this case, $e^{-\zeta'_\Gamma(0)}$ can be understood in terms of the so called Fuglede-Kadison determinant of the Laplacian on Γ.

Lemma

Let $\{\Gamma_n\}$ be a sequence of finite d-regular graphs, with Benjamini-Schramm limit Γ, satisfying

$$\limsup_n \Big(\int_0^{\varepsilon} \lambda^{-s}\, \mathrm{d}\mu_{\Gamma_n}(\lambda) \Big) \xrightarrow{\varepsilon \to 0} 0 \qquad (\star)$$

Then

1. *the zeta function converge:*

$$\frac{1}{|V(\Gamma_n)|} \zeta_{\Gamma_n}(s) \to \zeta_{\Gamma}(s)$$

2. *and*

$$t_{as}(\{\Gamma_n\}) = -\zeta'_{\Gamma}(0) \qquad (\star\star)$$

The sequence $\{B_{k,n}\}_n$ satisfies the condition $(\star)$.

Lyons proved $(\star\star)$ in the more general setting where the Benjamini-Schramm limit is a unimodular measure.

For De Bruijn graphs, the complexity was first computed by Strok (1992) using the Hutschenreuther statement:

$$t_{B_{k,n}} = \frac{1}{k^n}\chi'_{B_{k,n}}(2k) = (n+1)^{k-1}\prod_{i=1}^{n-1}(k^i(i+1))^{(k-1)^2 k^{n-i-1}}$$

where χ is the characteristic polynomial of the adjacency matrix.

Theorem

For $\Gamma = \mathrm{DL}(k,k) = \mathrm{Cay}(L_k, X)$, *we have*

$$-\zeta'_{L_k}(0) = t_{as} = \lim_{n\to\infty} \frac{\log(t_{B_{k,n}})}{k^n} = \dots$$

$$= \log(k) + (k-1)^2 \sum_{j\geq 2} \frac{\log(j)}{k^j} = \log(k) - (k-1)^2 \frac{d}{ds} \mathrm{Li}_s(\frac{1}{k})\Big|_{s=0},$$

where

$$\mathrm{Li}_s(z) = \sum_{j\geq 1} \frac{z^j}{j^s}$$

is the polylogarithm.

On the other hand, $-\zeta'_{L_k}(0) = \int_{\mathrm{Spec}(\Delta)} \log(\lambda)\,\mathrm{d}\mu_\Gamma =$

$$\log(k) + (k-1)^2 \sum_{q\geq 2} \frac{1}{k^q - 1} \Bigg(\sum_{\substack{1\leq p<q \\ (p,q)=1}} \log(2 - 2\cos(\frac{p}{q}\pi)) \Bigg)$$

Corollary

We have

$$-\frac{1}{2}\sum_{j\geq 2}\frac{\log(j)}{k^j} = \sum_{q\geq 2}\frac{1}{k^q-1}\Bigg(\sum_{\substack{1\leq p<q\\(p,q)=1}}\sum_{j\geq 1}\frac{\cos(\frac{p}{q}j\pi)}{j}\Bigg)$$