De Bruijn graphs, spider web graphs and Lamplighter groups

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Slides available on `www.leemann.website/slides/bruijn.pdf`

Goal

Explain links between some well-known graphs in order to better understand them:

- De Bruijn graphs (dynamical systems and combinatorics, computer science and bioinformatics),
- Spider web graphs (telephone switching networks, statistical physics),
- Schreier graphs of the Lampligther group (geometric group theory, spectrum of Cayley graph).

How it started

Theorem (Balram & Dhar, 2012)

Computation of the spectrum of the Spider web graphs $\mathcal{S}_{2,M,N}$.

"*In the limit of $M, N \to \infty$, the spectrum becomes purely discrete. This is very interesting, as the only other known example of a regular transitive infinite graph with a discrete spectrum of the laplacian is the Cayley graph of the lamplighter group, or its generalizations.*"

Question

Is the limit of the spectra of the $\mathcal{S}_{k,M,N}$ equal to the spectrum of the Lamplighter group?

Yes. And for good reasons.

Spider web (di)graphs

Let $k \geq 2$.

Definition

For all $M, N \in \mathbf{N}$, the spider-web digraph *is the labeled digraph $\vec{\mathcal{S}}_{N,M} = \vec{\mathcal{S}}_{k,N,M}$ with vertex set $\{0, \ldots, k-1\}^N \times M$.For every vertex $(x_1 \ldots x_N, i)$ and every $j \in \{0, \ldots, k-1\}$, there is a labeled arc*

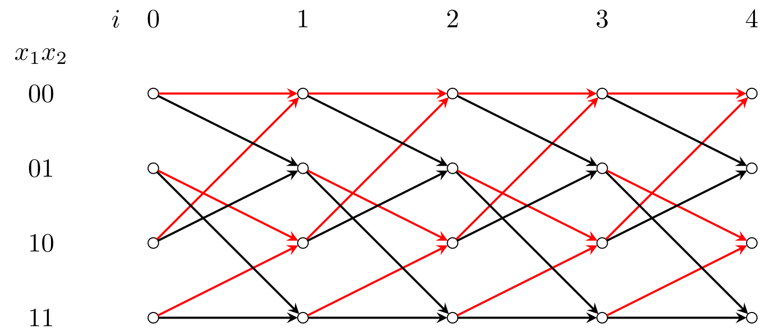
$$(x_1 \ldots x_N, i) \xrightarrow{j} (x_2 \ldots x_N j, i+1)$$

($i+1$ is taken modulo M).

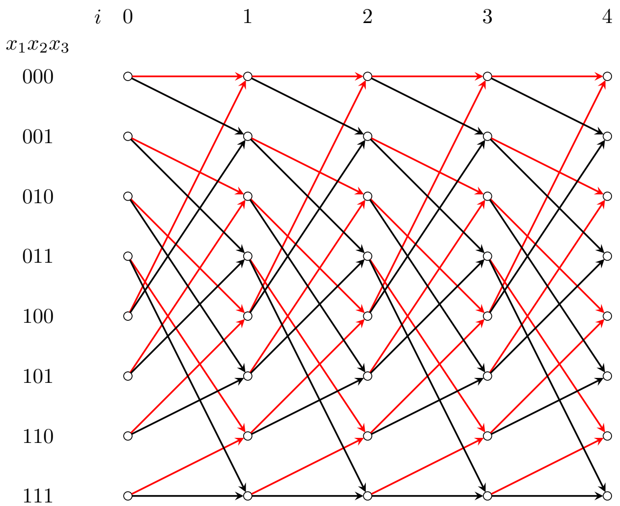
Definition

The spider-web graph *$\mathcal{S}_{N,M} = \mathcal{S}_{k,N,M}$ is the underlying graph of $\vec{\mathcal{S}}_{k,N,M}$.*

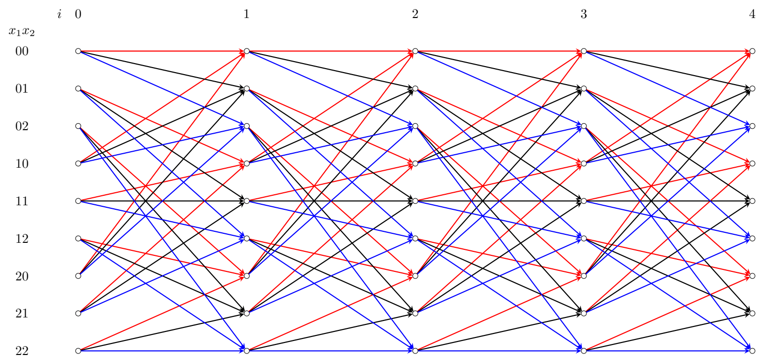
Example: $\vec{\mathcal{S}}_{2,2,4}$



Example: $\vec{\mathcal{S}}_{2,3,4}$



Example: $\vec{\mathcal{S}}_{3,2,4}$



The Lamplighter group

Definition

The Lamplighter group $\mathcal{L}_k$ *is the restricted wreath product*

$$\mathbf{Z}/k\mathbf{Z} \wr \mathbf{Z} = \Big(\bigoplus_{\mathbf{Z}} \mathbf{Z}/k\mathbf{Z}\Big) \rtimes \mathbf{Z}$$

where $\mathbf{Z}$ *acts on* $\bigoplus_{\mathbf{Z}} \mathbf{Z}/k\mathbf{Z}$ *by shifting the coordinate.*

We have

$$\mathcal{L}_K = \langle b, c \mid c^k, [c, b^n c b^{-n}]; n \in \mathbf{N}\rangle.$$

Generating sets and spectrum

Let $X_k = \{\bar{c}_i := bc^i\}_{i=0}^{k-1}$ and $Y_k = \{b, c\}$.

- Both X_k and Y_k generate $\mathcal{L}_k$,
- The spectrum of $\mathrm{Cay}(\mathcal{L}_k, X_k)$ is pure point [Grigorchuk & Zuk, 2001],
- The spectrum of $\mathrm{Cay}(\mathcal{L}_k, Y_k)$ contains no eigenvalue [Elek, 2003].

$\mathrm{Cay}(\mathcal{L}_k, X_k)$

- The graph $\mathrm{Cay}(\mathcal{L}_k, X_k)$ is isomorphic to the Diestel-Leader graph $\mathrm{DL}(k, k)$ (an horocyclic product of two $k+1$ regular tree).
- In fact, the group $\mathbf{Z}/k\mathbf{Z}$ can be replaced with any finite group of cardinality k.
- Vertices: $(\bigoplus_{\mathbf{Z}} \mathbf{Z}/k\mathbf{Z}) \rtimes \mathbf{Z}$
- Arcs

$$(\dots x_0 x_1 \dots x_i \dots, i) \xrightarrow{\bar{c}_j} (\dots x_0 x_1 \dots x_i (x_{i+1} + j) x_{i+1} \dots, i+1)$$

Main result

Theorem (GLS,2016)

The following diagram commutes, where the arrows stand for Benjamini-Schramm convergence of unlabeled graphs.

$$\begin{tikzcd}
\vec{\mathcal{S}}_{k,N,M} \arrow[r, "N \longrightarrow \infty"] \arrow[d, "M \downarrow \infty"'] \arrow[dr, "N, M \to \infty"] & \vec{\mathrm{Cay}}(\mathcal{L}_k, X_k) \arrow[d, equal] \\
\vec{\mathcal{S}}_{k,N,\infty} \arrow[r, "N \longrightarrow \infty"'] & \vec{\mathrm{Cay}}(\mathcal{L}_k, X_k)
\end{tikzcd}$$

Corollaries

Corollary

The following diagram commutes, where the arrows stand for Benjamini-Schramm convergence of unlabeled graphs.

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\mathcal{S}_{k,N,\infty} \arrow[r, "N \longrightarrow \infty"'] & \mathrm{Cay}(\mathcal{L}_k, X_k)
\end{tikzcd}$$

Corollaries

Corollary

The convergence of the graphs implies convergence of the spectral measure in the following sense:

$$\frac{1}{k^N \cdot M} \sum_{i=1}^{k^N \cdot M} \delta_{\lambda_i} \xrightarrow{n \to \infty} \mu_{\mathrm{Cay}(\mathcal{L}_k, X_k)}$$

where the λ_i are the eigenvalues of the Laplacian on $\mathcal{S}_{k,N,M}$ and the convergence is the weak convergence of measures.

Convergence of finite rooted graphs

Let $(\Gamma_n, v_n)_n$ be a sequence of finite rooted graphs of bounded degree.

- We say that a rooted graph (Γ, v) is the limit of $(\Gamma_n, v_n)_n$ if for every r, there exists N such that for all $n > N$, the ball of radius r in (Γ_n, v_n) is isomorphic to the ball of radius r in (Γ, v).
- The limit depends on the choice of the roots $v_n \in \Gamma_n$.
- Example: the cycles C_n tend to the biinfinite line $\mathbf{Z}$.

Convergence of finite graphs

Definition (Benjamini-Schramm)

Let $(\Gamma_n)_n$ be a sequence of finite graphs of bounded degree. One can consider them as rooted graphs by choosing a root in each Γ_n uniformly at random. This defines a sequence of probability measures on the space of (isomorphism classes of) rooted graphs, and one can consider its weak limit and call it the Benjamini-Schramm limit of the sequence $(\Gamma_n)_n$.

- The Benjamini-Schramm limit is a probability distribution on the space of rooted graphs, supported by the limits of the sequence of graphs $(\Gamma_n)_n$ for all possible choices of roots $v_n \in \Gamma_n$.
- In our case, the limit is the Dirac measure at $\mathrm{Cay}(\mathcal{L}_k, X_k)$.

Outline of the proof

Consider the de Bruijn digraphs $\vec{\mathcal{B}}_{k,N} \cong \vec{\mathcal{S}}_{k,N,1}$ $(M = 1)$.

1. $\vec{\mathcal{B}}_{k,N}$ is isomorphic to the Schreier graph of the action of $\mathcal{L}_k$ on the N^{th} level of a k-regular rooted tree.
2. These graphs converge to the Cayley graph $\vec{\text{Cay}}(\mathcal{L}_k, X_k)$,
3. It is enough to prove the convergence for $M = 1$.

De Bruijn (di)graphs

- An n-dimensional De Bruijn graph on k symbols, $B_{k,N}$, is a directed graph representing overlaps between sequences of symbols,
- Vertices: $\{0, 1, \ldots, k-1\}^N$,
- Arcs: for every $j \in \{0, \ldots, k-1\}$

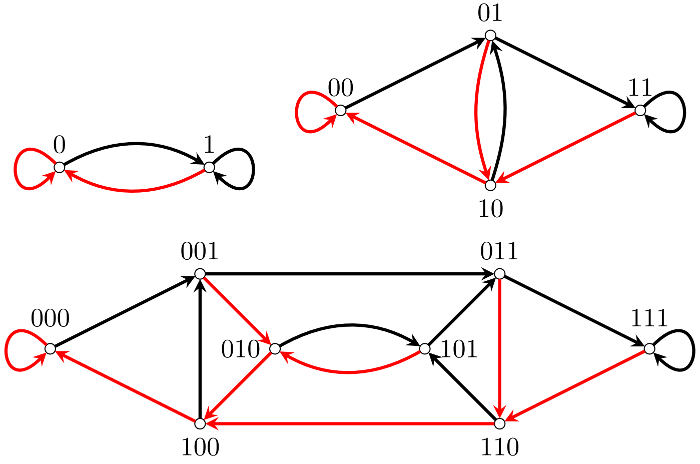
$$(x_1 \ldots x_N) \xrightarrow{j} (x_2 \ldots x_N j)$$

- $\vec{\mathcal{B}}_{k,N} \cong \vec{\mathcal{S}}_{k,N,1}$.

De Bruijn (di)graphs

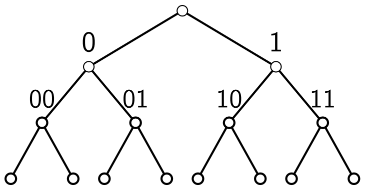
- De Bruijn graphs $\mathcal{B}_{k,n}$ are discrete models of the Bernoulli map $x \mapsto kx \pmod 1$ and therefore are of interest in the theory of dynamical systems.
- They also have applications in informatics (for peer-to-peer file sharing and parallel computing) and bioinformatics (genome assembly algorithms).

$\vec{\mathcal{B}}_{2,1}$, $\vec{\mathcal{B}}_{2,1}$ and $\vec{\mathcal{B}}_{2,3}$



The 2-regular rooted tree

$T_2 = \{0,1\}^*$

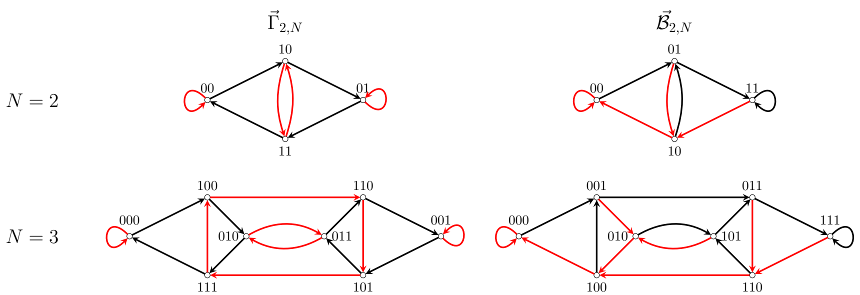


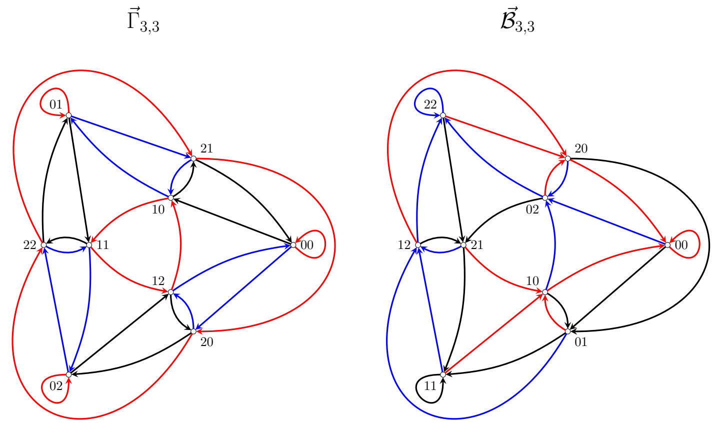
Action of $\mathcal{L}_k$ on a rooted tree

- The group $\mathcal{L}_k$ acts faithfully on T_k by

$$(x_1x_2x_3\dots).\bar{c}_r = ((x_1+r)(x_2+x_1)(x_3+x_2)\dots)$$

- The action is transitive on each levels,
- The graphs of the action, $\vec{\Gamma}_{k,N}$, look like the $\vec{\mathcal{B}}_{k,N}$.





Isomorphisms of graphs

Proposition (GLN,2016)

$\vec{\Gamma}_{k,N} \cong \vec{\mathcal{B}}_{k,N}$

Proof.

- $\vec{\Gamma}_{k,0} \cong \vec{\mathcal{B}}_{k,0}$ is the rose with k petals,
- $\vec{\Gamma}_{k,N+1}$ is the line graph of $\vec{\Gamma}_{k,N}$, (vertices are arcs of $\vec{\Gamma}_{k,N}$, arcs are succession of two consecutive arcs of $\vec{\Gamma}_{k,N}$),
- $\vec{\mathcal{B}}_{k,N+1}$ is the line graph of $\vec{\mathcal{B}}_{k,N}$.

□

Convergence

Proposition (G-Kravchenko, Pedro-Benjamin,GLN)

For all but countably many $\xi \in \partial T$*, the oriented graph* $\vec{\mathrm{Sch}}(\mathcal{L}, \mathrm{Stab}_{\mathcal{L}}(\xi), Y)$ *is isomorphic to* $\vec{\mathrm{Cay}}(\mathcal{L}, Y)$.

Corollary

$\vec{\mathcal{B}}_{k,N} \cong \vec{\Gamma}_{k,N} \xrightarrow{N\to\infty} \vec{\mathrm{Cay}}(\mathcal{L}_k, X_k)$ *and*

$\mathcal{B}_{k,N} \cong \Gamma_{k,N} \xrightarrow{N\to\infty} \mathrm{Cay}(\mathcal{L}_k, X_k)$.

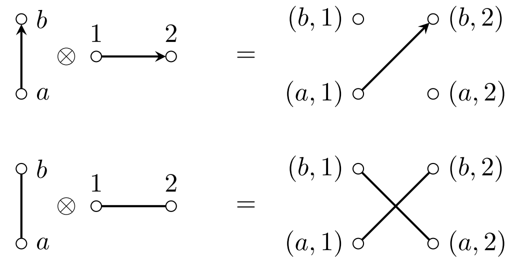
Tensor product

Definition

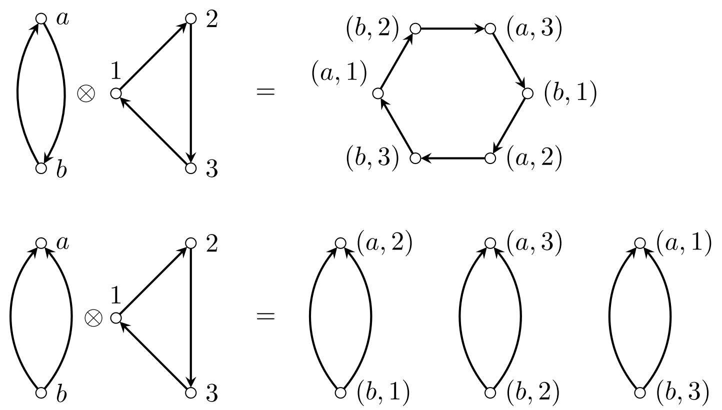
Let $\Gamma = (V, E)$ *and* $\Delta = (W, F)$ *be two digraphs. The tensor product* $\Gamma \otimes \Delta$ *is the digraph with vertices* $V \times W$ *and for every arcs* $v \to x$ *(in* Γ*) and* $w \to y$ *(in* Δ*) an arc* $(v, w) \to (x, y)$.

- This is the categorial product,
- Not necessarily connected,
- Depends on the orientation,
- $\vec{\mathcal{S}}_{k,N,M} \cong \vec{\mathcal{B}}_{k,N} \otimes \vec{C}_M$, where $\vec{C}_M$ is the oriented cycle of length M,
- $\mathcal{S}_{k,N,M} \ncong \mathcal{B}_{k,N} \otimes C_M$.

Examples of tensor products



Examples of tensor products



Tensor product and convergence

Theorem (GLN,16)

If $(\vec{\Gamma}_n, v_n)_n$ converges to $(\vec{\Gamma}, v)$ and $(\vec{\Theta}_m, y_m)_m$ converges to $(\vec{\Theta}, y)$ then the following diagram is commutative

$$\begin{tikzcd}
\left(\vec{\Gamma}_n \otimes \vec{\Theta}_m, (v_n, y_m)\right)^0 \arrow[r, "n \longrightarrow \infty"] \arrow[d, "m \downarrow \infty"'] \arrow[dr, "n,m \longrightarrow \infty"] & \left(\vec{\Gamma} \otimes \vec{\Theta}_m, (v, y_m)\right)^0 \arrow[d, "m \downarrow \infty"] \\
\left(\vec{\Gamma}_n \otimes \vec{\Theta}, (v_n, y)\right)^0 \arrow[r, "n \longrightarrow \infty"'] & \left(\vec{\Gamma} \otimes \vec{\Theta}, (v, y)\right)^0
\end{tikzcd}$$

- $\vec{C}_M \xrightarrow{M \to \infty} \vec{\mathbf{Z}}$
- All the connected components of $\vec{\mathrm{Cay}}(\mathcal{L}_k, X_k) \otimes \vec{C}_M$ and of $\vec{\mathrm{Cay}}(\mathcal{L}_k, X_k) \otimes \vec{\mathbf{Z}}$ are isomorphic to $\vec{\mathrm{Cay}}(\mathcal{L}_k, X_k)$,
- All the connected components of $\vec{\mathcal{B}}_{k,N} \otimes \vec{\mathbf{Z}}$ are isomorphic to $\vec{\mathcal{S}}_{k,N,\infty}$.

Theorem

$$\begin{tikzcd}
\left(\vec{\Gamma}_n \otimes \vec{\Theta}_m, (v_n, y_m)\right)^0 \arrow[r, "n \longrightarrow \infty"] \arrow[d, "m \downarrow \infty"'] \arrow[dr, "n,m \longrightarrow \infty"] & \left(\vec{\Gamma} \otimes \vec{\Theta}_m, (v, y_m)\right)^0 \arrow[d, "m \downarrow \infty"] \\
\left(\vec{\Gamma}_n \otimes \vec{\Theta}, (v_n, y)\right)^0 \arrow[r, "n \longrightarrow \infty"'] & \left(\vec{\Gamma} \otimes \vec{\Theta}, (v, y)\right)^0
\end{tikzcd}$$

Corollary

The following diagram commutes, where the arrows stand for Benjamini-Schramm convergence of unlabeled graphs.

$$\begin{tikzcd}
\vec{\mathcal{S}}_{k,N,M} \arrow[r, "N \longrightarrow \infty"] \arrow[d, "M \downarrow \infty"'] \arrow[dr, "N,M \to \infty"] & \vec{\mathrm{Cay}}(\mathcal{L}_k, X_k) \arrow[d, equal] \\
\vec{\mathcal{S}}_{k,N,\infty} \arrow[r, "N \longrightarrow \infty"'] & \vec{\mathrm{Cay}}(\mathcal{L}_k, X_k)
\end{tikzcd}$$

Some consequences 1

- In 1998 Dellorme and Tillich proved that $\mathcal{B}_{k,N}$ is cospectral with a disjoint union of (weighted) loops and paths,
- Using the tensor product with $\vec{C}_M$, it is easy to extend this result to $\mathcal{S}_{k,N,M}$ and compute its spectrum,
- Using the convergence, we recover

$$\mu_{\mathrm{Cay}(\mathcal{L}_k, X)} = (k-1)^2 \sum_{q \geq 2} \frac{1}{k^q - 1} \Bigg(\sum_{\substack{1 \leq p < q \\ (p,q)=1}} \delta_{2k(1-\cos(\frac{p}{q}\pi))} \Bigg).$$

Some consequences 2

- LN Computation of the complexity (number of covering trees) $t(\mathcal{S}_{k,N,M})$ (Stok, 1992 for de Bruijn graphs),
- Let $\Gamma = \mathrm{Cay}(\mathcal{L}_k, X_k)$ and let $p_d(o;\Gamma)$ denotes the probability that the simple random walk started at o is back at o after d steps. By a general result of Lyons (2003):

$$\sum_{j\geq 1} \frac{1}{j} p_j(o;\Gamma) = \log(2k) - \lim_N \frac{\log(t(\mathcal{B}_{k,N}))}{k^N}$$

- LN Computation of the spectral zeta function for $\Gamma = \mathrm{Cay}(\mathcal{L}_k, X_k)$

$$\zeta_\Gamma(s) = \int_{\mathrm{Spec}(\Gamma)} \lambda^{-s} \mathrm{d}\mu_\Gamma(\lambda),$$

  related to the determinant of the Laplacian by $\det \Delta_\Gamma = e^{-\zeta'_\Gamma(0)}$.
- ...

Generalizations

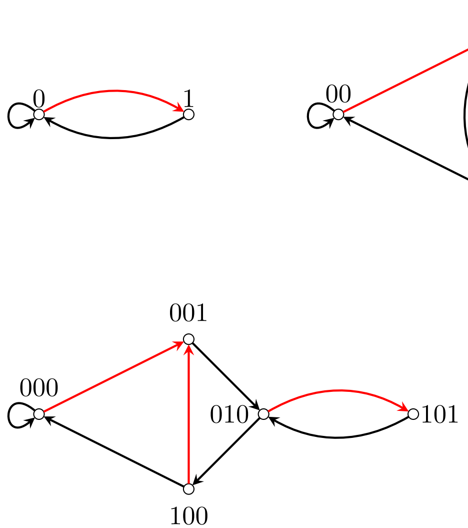
De Bruijn graphs correspond to the full shift on $\{0, 1, \dots, k-1\}^{\mathbf{Z}}$. What happens if we take only a subshift?

Definition

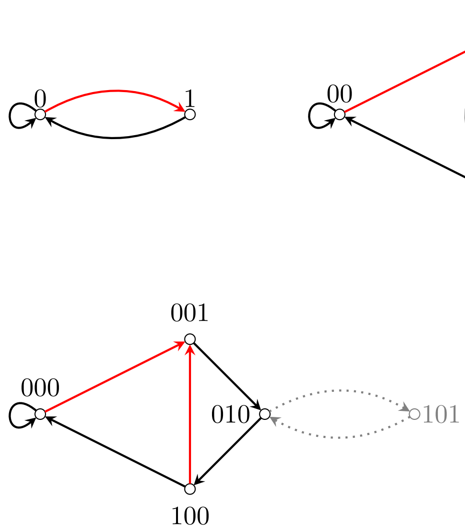
Let Σ be a subshift (closed invariant subset) of $\{0, 1, \dots, k-1\}^{\mathbf{Z}}$.
The Rauzy digraphs $\vec{R}_{k,\Sigma,N}$ is the digraph with
vertices: admissible words of length N
arcs: $x_1 \dots x_N \xrightarrow{x_{N+1}} x_2 \dots x_N x_{N+1}$ if $x_1 \dots x_N x_{N+1}$ is admissible.

As a variation, one can look at $\vec{R}_{\circlearrowright,k,\Sigma,N}$, where vertices are supposed to by cyclically admissible.

The Fibonacci subshift: $\vec{R}_{2,\{11\},N}$



The Fibonacci subshift: $\vec{R}_{\circlearrowright,2,\{11\},N}$



Convergence of Rauzy graphs

Proposition (L.)

Let $\Sigma \leq \{0, 1, \ldots, k-1\}^{\mathbf{Z}}$ be an irreducible and weakly aperiodic subshift of finite type. Then the limit of $(\vec{R}_{k,\Sigma,N})_N$ is supported on horocyclic products of trees.

- Observe that Σ is irreducible if and only if all the $\vec{R}_{k,\Sigma,N}$ are strongly connected,
- Is it possible to better understand the measure (not only its support)?
- Yes. Ongoing project with V. Kaimanovich and T. Nagnibeda

Digraph structure on Σ

We endow Σ with a digraph structure in the following way.

- Vertices: Σ,
- Arcs: there is an arc from ω to ω' if and only if ω' is obtained by shifting ω by 1 and possibly changing the value of ω at 0.

Define $g\colon \Sigma \to \{\text{rooted graphs}\}$ by sending ω to the connected component of the graph Σ containing ω, rooted at ω.

Convergence of Rauzy graphs

Let Σ be an irreducible subshift of finite type. There exists a unique invariant measure μ that maximizes the Kolmogorov-Sinai entropy (μ is the limit of the uniform measures on cylinders)

Theorem (KLN,21$^+$)

Let Σ be an irreducible weakly aperiodic subshift of finite type. Then the digraphs $\vec{R}_{\circlearrowleft,k,\Sigma,N}$ converge to $g_*(\mu)$.

Similar result for the convergence of $\vec{R}_{k,\Sigma,N}$.