On the structure of finitely generated subgroups of branch groups

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Motivations

- What are branch groups, and why we would like to study them;
- Profinite topology and finitely generated subgroups;
- Consequences.

Branch groups

- Groups with a rich subnormal subgroup structure ($H = H_0 \trianglelefteq H_1 \trianglelefteq \cdots \trianglelefteq H_n = G$);
- Contains example of groups with exotic properties:
  - Finitely generated infinite torsion groups,
  - Groups of intermediate growth,
  - Amenable but non-elementary amenable groups;
- Naturally appears in the classification of just infinite groups (infinite groups whose proper quotients are finite);
- Admits a nice action on a rooted tree (and hence are residually finite);
- Many branch groups are recursively presented (a weaker notion than finitely generated).

The profinite topology

Any group G can be endowed with the profinite topology: the topology generated by all left cosets of finite index normal subgroups.

- G is residually finite if and only if $\{1\}$ is closed (for finitely generated groups: if and only if G acts faithfully on a locally finite rooted tree);
- If G is residually finite, recursively presented and has an algorithm listing all finite index normal subgroups, then the word problem is solvable.

Finitely generated branch groups with the *congruence subgroup property* have an algorithm listing all finite index normal subgroups.

Profinite topology: beyond residually finite groups

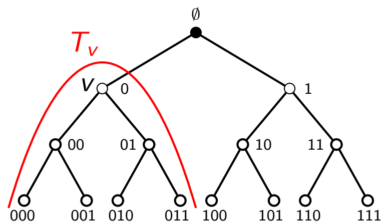
- A group G is LERF (locally extensively residually finite, or subgroup separable) if every finitely generated subgroup is closed in the profinite topology;
  - If G is LERF, recursively presented and has an algorithm listing all finite index normal subgroups, then the generalized word problem is solvable (subgroup membership for finitely generated subgroups),
  - Finite groups, f. g. abelian groups, virtually polycyclic groups, f. g. free groups, surface groups, limit groups, free metabelian groups, some right-angled Artin groups are LERF. Some branch groups are LERF.
- A group G has the Ribes-Zalesskiĭ property if for any n and every finitely generated subgroups $H_1, \dots, H_n$, the subset $H_1 H_2 \cdots H_n$ is closed;
  - $n = 0$ is residually finite, $n = 1$ is LERF,
  - Finite groups, f. g. abelian groups, f.g. free groups, limit groups, Kleinian groups and a few additional examples have the RZ property. Conjecture: some branch groups have the Ribes-Zalesskiĭ property.

Classification of finitely generated subgroups: other consequences

- Can be used to describe weakly maximal subgroups (subgroups maximal among infinite index subgroups);
- G has the Howson property if for H, K finitely generated subgroups, $H \cap K$ is still finitely generated.
  - Most finitely generated branch groups do not have the Howson property [Francoeur 2025+]. But one hope to have an algorithm to decide whenever $H \cap K$ is finitely generated or not.

Regular rooted trees

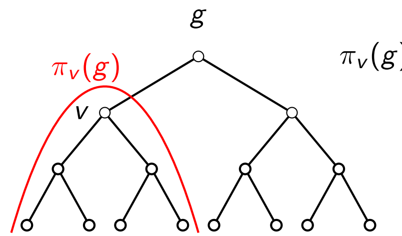
- $T = T_d$: the d-regular rooted tree (the root has degree d and each other vertex has degree $d+1$);



- Vertices of T_d are in bijection with finite words on the alphabet $\{0, \dots, d-1\}$ (root $\leftrightarrow \emptyset$ the empty word);
- The n^{th} level $\mathcal{L}_n$ of the tree is the set of vertices at distance n of the root;
- T_v is the subtree of T consisting of vertices below v.

Sections of elements of Aut(T)

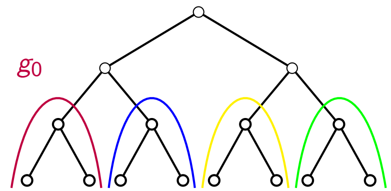
- For v a vertex of T and $g \in \mathrm{Stab}_{\mathrm{Aut}(T)}(v)$, the section $\pi_v(g) = g_{|_v}$ of g at v is the automorphism of T_v induced by g.



$$\pi_v(g) \in \mathrm{Aut}(T_v) \cong \mathrm{Aut}(T)$$

- Elements g that fixe $\mathcal{L}_n$ are usually described as the product of their sections:

$$g = (g_0, g_1, g_2, g_3)$$



Self-similar groups

Definition
A group $G \leq \mathrm{Aut}(T)$ is self-similar if for every vertex v in T we have $\pi_v(\mathrm{Stab}_G(v)) \leq G$.

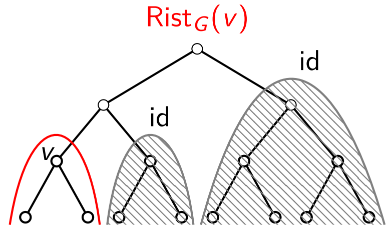
Definition
A group $G \leq \mathrm{Aut}(T)$ is self-replicating (or fractal) if for every vertex v in T we have $\pi_v(\mathrm{Stab}_G(v)) = G$.

Some subgroups of $\mathrm{Aut}(T_d)$

Let $G \leq \mathrm{Aut}(T_d)$. The following subgroups play an important role:

- Stabilizers of vertices $\mathrm{Stab}_G(v)$;
- Pointwise stabilizers of levels $\mathrm{Stab}_G(\mathcal{L}_n)$;
- Rigid stabilizer of vertices:

$$\mathrm{Rist}_G(v) := \{g \in G \mid g \text{ acts trivially outside } T_v\} = \bigcap_{w \notin T_v} \mathrm{Stab}_G(w)$$



Some subgroups of $\mathrm{Aut}(T_d)$

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- Stabilizers of vertices $\mathrm{Stab}_G(v)$;
- Pointwise stabilizers of levels $\mathrm{Stab}_G(\mathcal{L}_n)$;
- Rigid stabilizer of vertices $\mathrm{Rist}_G(v)$,
- Rigid stabilizer of levels: $\mathrm{Rist}_G(\mathcal{L}_n) := \prod_{v \in \mathcal{L}_n} \mathrm{Rist}_G(v)$.
  Carefull: $\mathrm{Rist}_G(\mathcal{L}_n) \neq \mathrm{Rist}_{\mathrm{Aut}(T)}(\mathcal{L}_n) \cap G$.

The congruence subgroup property

Definition
Let T be a rooted tree. A subgroup $G \leq \mathrm{Aut}(T)$ has the congruence subgroup property if for any finite index subgroup H there exists n such that $\mathrm{Stab}_G(n) \leq H$

G has the congruence subgroup property if and only if the profinite topology (generated by the finite index normal subgroups and their cosets) and the *natural topology on* $\mathrm{Aut}(T)$ (generated by the $\mathrm{Stab}_G(n)$ and their cosets) agree.
The congruence subgroup property provides an algorithm describing finite index normal subgroups of G. If G is recursively presented, then the word problem is solvable (and the generalized word problem also, provided that G is LERF).

Branch groups

Definition

A subgroup G of $\mathrm{Aut}(T)$ is branch if for all n

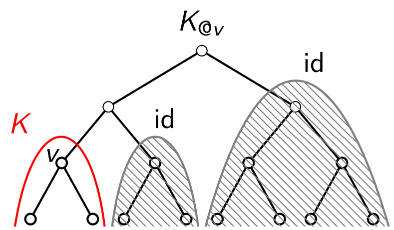
1. G acts transitively on $\mathcal{L}_n$,
2. $\mathrm{Rist}_G(\mathcal{L}_n)$ is a finite index subgroup of G.

Regularly branch groups

Definition

A self-similar subgroup G of $\mathrm{Aut}(T)$ is regularly branch if

1. G acts transitively on $\mathcal{L}_n$ for every n;
2. There exists a finite index subgroup $K \leq G$ such that $K_{@v} \leq K$ for every vertex v, where $K_{@v} := \{g \in \mathrm{Rist}(v) \mid \pi_v(g) \in K\}$.



Regularly branch groups are branch.

Regularly branch groups

Example

The first Grigorchuk group $\mathfrak{G}$, the Gupta-Sidki p-groups ($p \geq 3$ prime), torsion GGS groups (acting on T_p, $p \geq 3$ prime).

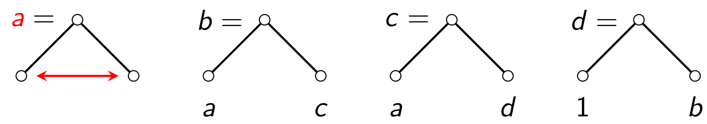
All these examples:

- Are regularly branch;
- Have the congruence subgroup property;
- Are recursively presented;
- Are infinite, just infinite, torsion, all their maximal subgroups are of finite index, ...

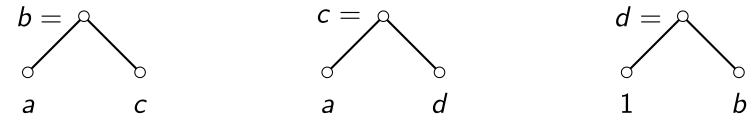
Moreover $\mathfrak{G}$ has intermediate growth, ...

The first Grigorchuk group

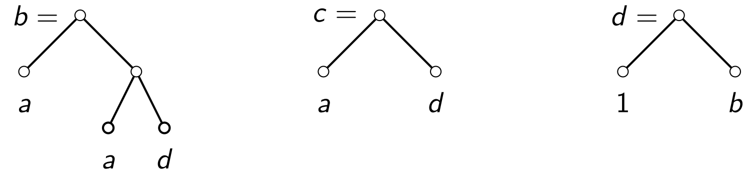
The first Grigorchuk group $\mathfrak{G} = \langle a, b, c, d \rangle$ acts on T_2 and is generated by



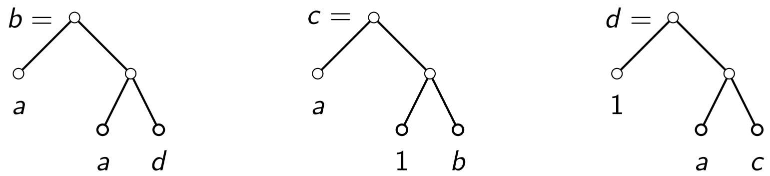
The first Grigorchuk group



The first Grigorchuk group

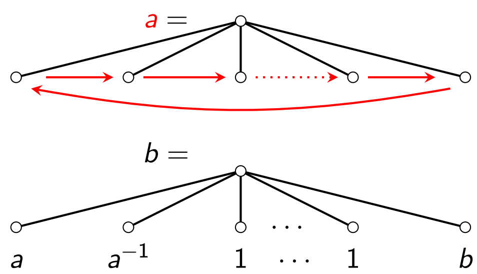


The first Grigorchuk group



The Gupta-Sidki p-group

The group G_p acts on T_p ($p \geq 3$ prime) and is generated by a and b, where



GGS groups

- Let $p \geq 3$ be a prime and let $\mathbf{e} = (e_0, \ldots, e_{p-2})$ be a vector in $(\mathbf{Z}/p\mathbf{Z})^{p-1} \setminus \{0\}$. The GGS group $G_{\mathbf{e}} = \langle a, b \rangle$ with defining vector $\mathbf{e}$ is the subgroup of $\mathrm{Aut}(T_p)$ generated by

$$a = \text{cyclic permutation } (12\ldots p) \text{ of the first level vertices}$$
$$b = (a^{e_0}, \ldots, a^{e_{p-2}}, b);$$

- The group $G_{\mathbf{e}}$ is torsion if and only if $\sum_{i=0}^{p-2} e_i = 0$;
- The Gupta-Sidki p-group correspond to the special case $\mathbf{e} = (1, -1, 0, \ldots, 0)$.

Some finitely generated subgroups

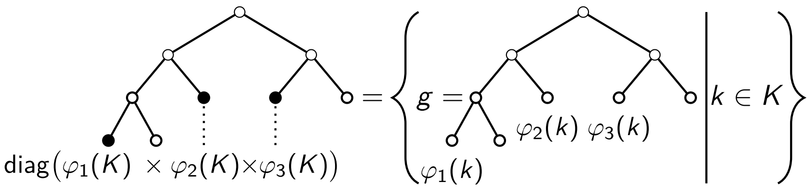
Let $G \leq \mathrm{Aut}(T)$ be a finitely generated group regularly branch over K. Then the following are finitely generated subgroups of G:

- K;
- $K_{@v}$ for every vertex v.

One can use the above to construct other finitely generated subgroups.

Diagonal subgroups

Let $G \leq \mathrm{Aut}(T)$ be a finitely generated group regularly branch over K. Let $v_1, \ldots, v_n$ be pairwise incomparable vertices and $\varphi_1, \ldots, \varphi_n$ be automorphisms of K. This datas define a diagonal subgroup (over K):



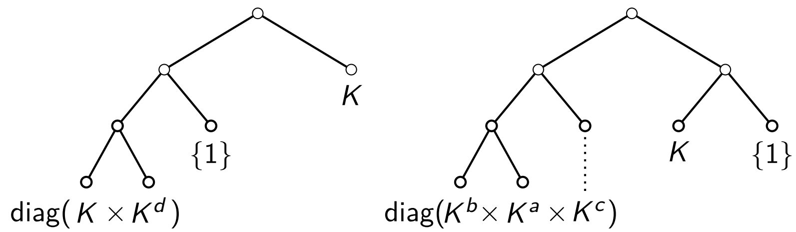
Diagonal subgroups are finitely generated.

Block subgroups

Definition

A block subgroup (over K) is a finite product of diagonal subgroups over K (such that all the corresponding vertices are incomparable).

Example



Block subgroups are finitely generated, as well as virtually block subgroups ($H \leq G$ such that there exists $B \leq H$ of finite index which is block).

Main result

Main theorem

Let G be either the first Grigorchuk group or a torsion GGS group (for some prime p). Then every finitely generated subgroup of G is virtually a block subgroup over K.

For the first Grigorchuk group, $K = \langle [a, b] \rangle^{\mathfrak{G}} \leq_{16} \mathfrak{G}$. For a GGS group $G = G_{\mathbf{e}}$, $K = \gamma_3(G) = \langle [G', G] \rangle \leq_{p^3} G$ if $\mathbf{e}$ is symmetric, or over $K = G' \leq_{p^2} G$ if $\mathbf{e}$ is not symmetric.

Abstract version

Let G be a regularly branch group and let RB be the set of subgroups on which G regularly branch. RB admits a unique maximal element, which is called the maximal branching subgroup of G.

Theorem

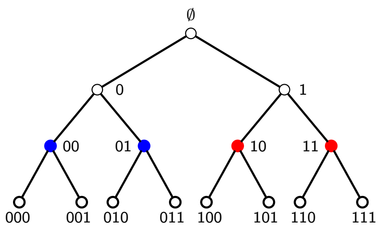
Let $G \leq \mathrm{Aut}(T)$ be a finitely generated, self-replicating regularly branch group and let K be its maximal branching subgroup. Suppose that:

- *G acts tree-primitively on T;*
- *G has trivial branch kernel;*
- *G has the subgroup induction property.*

Then every finitely generated subgroup of G is virtually a block subgroup over K.

Tree-primitive action

The action of a subgroup $G \leq \mathrm{Aut}(T)$ on T needs to preserve the tree structure and therefore cannot be primitive, even when restricted to some level $\mathcal{L}_n$.
For example, if $T = T_2$, then the partition $\{\{00, 01\}, \{10, 11\}\}$ of $\mathcal{L}_2$ is necessarily preserved by G.



Definition

The action of G on T is tree-primitive if for every n, the only partitions of $\mathcal{L}_n$ preserved by the action of G are the partition that are already preserved by $\mathrm{Aut}(T)$.

Trivial branch kernel

Definition

A branch group $G \leq \mathrm{Aut}(T)$ is said to have trivial branch kernel if for every normal subgroup $N \trianglelefteq G$ of finite index, there exists n such that $\mathrm{Rist}_G(n) \leq N$.

A branch group G can be endowed with the profinite topology (generated by the finite index normal subgroups and their subgroups), but also with the branch topology (generated by the $\mathrm{Rist}_G(n)$ and their cosets). G has trivial branch kernel if and only if these two topologies coincide. Branch group with the congruence subgroup property have trivial branch kernel.

The subgroup induction property

Definition
Let $G \leq \mathrm{Aut}(T)$ be a self-similar group. A family $\mathcal{C}$ of subgroups of G is said to be inductive if

1. Both $\{1\}$ and G belong to X,
2. If $H \in \mathcal{C}$ and L contains H as a finite index subgroup, then $L \in \mathcal{C}$,
3. If H is a finitely generated subgroup of $\mathrm{Stab}_G(\mathcal{L}_1)$ and all first level sections of H are in $\mathcal{C}$, then H is in $\mathcal{C}$.

It is clear that the collection $\mathcal{C}$ of finitely generated subgroups of G is inductive.

Definition (Grigorchuk-Wilson, 2003)
A self-similar group G has the subgroup induction property (SIP for short) if for any inductive class of subgroups $\mathcal{C}$, each finitely generated subgroup of G is contained in $\mathcal{C}$.

The subgroup induction property: consequences

Theorem (Francoeur-L., 2025, (Grigorchuk-Wilson 2003, Garrido 2016))
Let G be a finitely generated branch group with the subgroup induction property. Then, G is torsion and just infinite.

1. *If $G \leq \mathrm{Aut}(T_d)$ is self-replicating and $H \leq G$ is finitely generated, then H is commensurable with one of $1, G, G^2, \ldots, G^{d-1}$;*
2. *If $G \leq \mathrm{Aut}(T_p)$ with p prime, then every maximal subgroup of G is of finite index;*
3. *If G is self-similar, then it is LERF;*
4. *If $G \leq \mathrm{Aut}(T_p)$ with p prime is self-similar and $H \leq G$ is finitely generated, then every maximal subgroup of H is of finite index in H and every weakly maximal subgroup of H is closed in the profinite topology.*

The subgroup induction property: examples

The following groups have the subgroup induction property:

- The first Grigorchuk group [Grigorchuk-Wilson, 2003];
- The Gupta-Sidki 3-group [Garrido, 2016];
- Torsion GGS groups (for a prime p) [Francoeur-L., 2025].

Consequences of the main result

Let G be either the first Grigorchuk group or torsion GGS group (for some prime p). Then;

- G is LERF (alternative proof);
- We have a classification of weakly maximal subgroups of G [L. 2025];
- We have a classification of finitely generated self-commensurating subgroups of G (can be used to construct irreducible representations of G) [Francoeur-L.-Nagnibeda 2025^{++}].

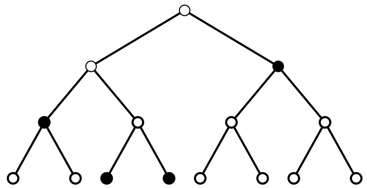
Some projects

Use Theorem A to:

- show that $\mathfrak{G}$ has the Ribes-Zalesskiĭ property;
- have a algorithm that given two finitely generated subgroups H and K of $\mathfrak{G}$ decide whenever $H \cap K$ is finitely generated or not.

A few words on the proof

A subgroup $H \leq G \leq \mathrm{Aut}(T)$ is fully supported if there exists a transversal $V \subseteq T$ such that for all $v \in V$, $\pi_v(\mathrm{Stab}_H(v))$ is finite or $= G$.



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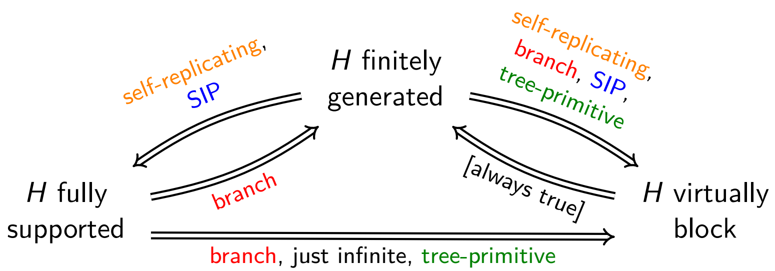
Let $G \leq \mathrm{Aut}(X^*)$ be a finitely generated group and $H \leq G$ a subgroup. Then we have the following implications:



Virtually block implies finitely generated

Since G finitely generated, so is K as well as any block subgroup. Indeed, blocks subgroups are isomorphic to K^n for some n. Therefore, all virtually block subgroups are finitely generated.

Finitely generated implies fully supported

Let $G \leq \mathrm{Aut}(T)$ be a self-replicating subgroup. One easily show that the class $\mathcal{C}$ of finitely generated fully supported subgroups of G is inductive. By the subgroup induction property, we conclude that $\mathcal{C}$ is the whole class of finitely generated subgroups.

Fully supported implies virtually block

Let H be a fully supported subgroup and let V be the transversal witnessing it. Let $F \subseteq V$ be the subset of vertices such that $\pi_v(\mathrm{Stab}_H(v)) = G$.
For any $v \in F$, one can find a *minimal dependance set* $W \subseteq F$ such that

$$\pi_v\left(\mathrm{Stab}(W) \cap \prod_{w \in W} \mathrm{Rist}_H(w)\right) \neq \{1\}.$$

One can shows that: W does not depend on v. Moreover, if G is branch, just infinite and with a tree primitive action, and $W = \{w_1, \dots, w_n\}$ is a minimal dependance set, then they exists $Z = \{z_1 \leq w_1, \dots, z_n \leq w_n\}$ such that $\mathrm{Stab}(Z) \cap \prod_{z \in Z} \mathrm{Rist}_H(z)$ is a diagonal subgroup.

Some elements of the proof

A throughout study of almost normal subgroups: $H \leq G$ such that H is normal in a finite index subgroup of G.

Definition

Let $H \leq G$ be an almost normal subgroup. An almost normal subgroup $K \leq G$ is a complement for H if: $H \cap K = \{1\}$, $[H, K] = \{1\}$ (i.e. HK is the direct product of H and K) and HK has finite index in G.

Proposition

Let G be a just infinite branch group and let $H \leq G$ be an almost normal subgroup. Then H admits an almost normal complement.

More groups satisfying the hypothesis of Theorem A?

Want G finitely generated, self-replicating, regularly branch, with trivial branch kernel, a tree-primitive action and the subgroup induction property.
Many examples of finitely generated, self-replicating, regularly branch groups with the congruence subgroup property (and hence with trivial branch kernel) are known.
It remains to show that some of these examples have a tree-primitive action and the subgroup induction property.

Checking for tree-primitive actions

Tree-primitivity is about the action on the whole tree. Hopefully, it is enough to check it on the first two levels only.

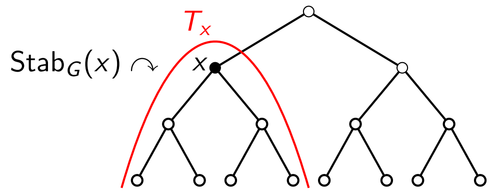
Theorem

Let G be a self-replicating group acting spherically transitively on T such that the action of G the first level $\mathcal{L}_1$ is primitive. Suppose moreover that:

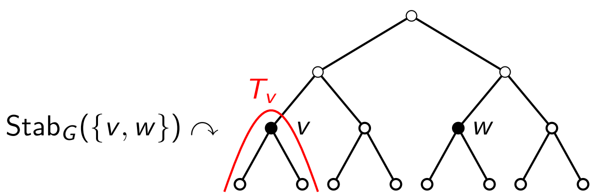
1. $\mathrm{Stab}_G(\mathcal{L}_1)$ *acts spherically transitively on* T_x *for all* $x \in \mathcal{L}_1$*;*
2. *For every* $v, w \in \mathcal{L}_2$*, the subgroup* $\mathrm{Stab}_G(\{v, w\})$ *acts spherically transitively on* T_v*;*
3. *There exists* $x_0 \neq y_0 \in \mathcal{L}_1$ *such that for every* $v \in T_{x_0} \cap \mathcal{L}_2$ *and* $w \in T_{y_0} \cap \mathcal{L}_2$*, we have* $\mathrm{Stab}_G(v) \cap \mathrm{Stab}_G(y_0) \not\leq \mathrm{Stab}_G(w)$*.*

Then the action of G on X is tree-primitive.

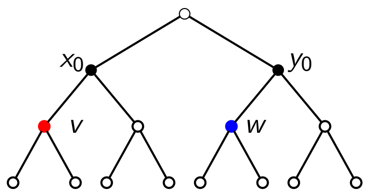
1. $\mathrm{Stab}_G(\mathcal{L}_1)$ acts spherically transitively on T_x for all $x \in \mathcal{L}_1$;



2. For every $v, w \in \mathcal{L}_2$, the subgroup $\mathrm{Stab}_G(\{v, w\})$ acts spherically transitively on T_v;



There exists $x_0 \neq y_0 \in \mathcal{L}_1$ such that for every $v \in T_{x_0} \cap \mathcal{L}_2$ and $w \in T_{y_0} \cap \mathcal{L}_2$, we have $\mathrm{Stab}_G(v) \cap \mathrm{Stab}_G(y_0) \not\leq \mathrm{Stab}_G(w)$.



There exists g fixing y_0 and v, but moving w.

Checking for the subgroup induction property

Definition
Let $G \leq \mathrm{Aut}(T)$ be a self-similar group. A family $\mathcal{C}$ of subgroups of G is said to be strongly inductive if

1. Both $\{1\}$ and G belong to X,
2. If L contains H as a finite index subgroup, then $L \in \mathcal{C} \iff H \in \mathcal{C}$,
3. If H is a finitely generated subgroup of $\mathrm{Stab}_G(\mathcal{L}_1)$ and all first level sections of H are in $\mathcal{C}$, then H is in $\mathcal{C}$.

Definition
A self-similar group G has the weak subgroup induction property (wSIP for short) if for any strongly inductive class of subgroups $\mathcal{C}$, each finitely generated subgroup of G is contained in $\mathcal{C}$.

It is clear that SIP $\implies$ wSIP.

The weak subgroup induction property

Proposition (Francoeur-L. 2025)
Let G be a self-replicating group with the congruence subgroup property and such that for every $v \in \mathcal{L}_n$, $\pi_v(\mathrm{Stab}_G(\mathcal{L}_n)) = G$. Then if G has wSIP it also has SIP.

Proposition (Francoeur-L. 2025)
Let G be a finitely generated self-similar group with SIP. Then for every $v \in \mathcal{L}_1$, $\pi_v(\mathrm{Stab}_G(\mathcal{L}_n)) = G$.

Proposition (Grigorchuk-L.-Nagnibeda 2021)
Let G be a self-replicating group. Then G has the wSIP if and only if: for every finitely generated $H \leq G$, there is a transversal V of T such that for every $v \in V$ the section $\pi_v(\mathrm{Stab}_H(V))$ is either trivial or of finite index in G.

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