

# Closed subgroups and Cayley graphs of infinite groups

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My research interests lie in geometric and combinatorial group theory, as well as in symbolic dynamic. Graphs play an important role in my works and projects. I am particularly interested in groups acting on graphs (for examples on rooted trees) and their closed subgroups and on graphs associated with groups actions (Cayley and Schreier graphs) and rigidity phenomenon associated to them.

Below I will develop two specific points.

## 1 Closed subgroups of branch groups

### 1.1 Current state of research in the field

This project is at the intersections of three important themes in geometric group theory: branch groups, self-similar groups and finally the profinite topology and the (locally) extended residually finite groups (i.e. (L)ERF groups).

A group is *just infinite* if it is infinite and all its proper quotients are finite. For an infinite group, this is a natural relax of the simplicity condition: we allow  $G$  to have non-trivial quotients, but only if they are finite, and hence in some sense trivial from the geometric group theory point of view. This condition is both natural from the large-scale geometry point of view, and useful in practice as every finitely generated infinite group admits a just infinite quotient. A major discovery of Wilson in the 70' (later refined by Grigorchuk) is that the class of just infinite groups splits into three subclasses which roughly are infinite simple groups, hereditarily just infinite groups (every finite index subgroup of  $G$  is also just infinite) and just infinite branch groups. Groups in the last category are examples of groups acting faithfully on a rooted tree and are characterized by a rich structures of subgroups, despite being just infinite. One striking example of such a group is the first Grigorchuk group  $\mathcal{G}$ , which was the first example of a group of intermediate growth, and hence also the first example of an amenable but not elementary amenable group [17, 18]. On the other hand, self-similar groups naturally

appear in dynamics [35] and are examples of automata groups. Although quite different, these two classes of groups have a large intersection, and many self-similar groups are also branch. In the class of finitely generated branch self-similar groups, there are torsion groups and torsion free groups; groups of intermediate growth and groups of exponential growth; amenable and nonamenable groups. Self-similar branch groups are a great source of examples and counter-examples in group theory and the study of these groups and their subgroup structure as attired a lot of attention among geometric group theorist these last years.

Another interesting subject in geometric group theory is the profinite topology. That is, the topology on  $G$  generated by the finite index subgroups (and their left cosets). A group  $G$  is *residually finite* if the trivial subgroup is closed in the profinite topology. Residually finite groups are a natural generalization of finite groups, and hence are an interesting object of studies. Here are some of their many characterizations.

**Fact 1.** *For a group  $G$ , the following are equivalent*

1.  $G$  is residually finite,
2. the profinite topology on  $G$  is Hausdorff,
3. the intersection of normal subgroups of finite index of  $G$  is trivial,
4. For every  $g \neq 1$ , there exists a finite quotient  $\pi: G \rightarrow H$  such that  $\pi(g) \neq 1$ ,
5.  $G$  embeds in its profinite completion,
6. (if  $G$  is finitely generated)  $G$  acts faithfully on a rooted tree.

By definition, for every group  $G$ , finite index subgroups are profinite-closed (i.e. closed in the profinite topology), and if  $G$  is residually finite, then finite subgroups are also profinite closed. Given a group  $G$ , one can then ask the following

**Question 2.** *What are the profinite closed subgroups of  $G$ ?*

If every subgroup of  $G$  is profinite closed, we will say that  $G$  is *extended residually finite* (ERF), while  $G$  is called *locally extended residually finite* (LERF) if all its finitely generated subgroups are profinite-closed.

While the class of residually finite groups contains a lot of well-known examples and has many distinct characterizations, the classes of LERF and ERF subgroups are less understood. As examples of ERF groups we have finite groups, finitely generated abelian groups, and virtually polycyclic groups [30]. The class of LERF groups contains all ERF groups, as well as finitely generated free groups, surface groups and more generally limits

groups [48]. Another interesting properties of finitely generated free groups is that a product  $\prod_{i=1}^n H_i$  of finitely many finitely generated subgroups is closed. This remarkable property is known (for finitely generated groups) only for free groups and some straightforward generalizations.

Since branch groups are by definition subgroups of  $\text{Aut}(T)$ , where  $T$  is a rooted tree, they are residually finite. One can hence ask the following.

**Question 3.** *Is the first Grigorchuk group  $\mathcal{G}$  ERF? If not, is it possible to describe all its profinite-closed subgroups?*

**Question 4** ([21]). *Let  $H_1$  to  $H_n$  be finitely generated subgroups of  $\mathcal{G}$ . Is the subset  $\prod_{i=1}^n H_i$  profinite closed?*

And more generally

**Question 5.** *Which branch groups are LERF? and ERF?*

Some partial results to the above question are already known. For example, all maximal subgroups of  $\mathcal{G}$  are of finite index [37] and hence profinite closed. By [20], this implies that the weakly maximal subgroups of  $\mathcal{G}$  (subgroups maximal among infinite index subgroups) are also closed. Finally, it follows from [20] that  $\mathcal{G}$  is LERF.

Finally, an important property in the context of closed subgroups of  $G \leq \text{Aut}(T)$  is the so-called *congruence subgroup property* which asserts that the profinite topology on  $G$  coincide with the restriction to  $G$  of the natural topology of  $\text{Aut}(T)$  (i.e. the topology generated by the pointwise stabilizers of the levels of the tree). Indeed, if  $G \leq \text{Aut}(T)$  has the congruence subgroup property, then in order to understand the profinite topology of  $G$  we don't need to look at all finite index subgroups, but it is enough to look at the stabilizers of the levels and we can hence more geometric methods. Due to its importance, this property has been extensively investigated these last twenty years, see [5, 14] and the references therein. In particular, it is known that  $\mathcal{G}$  as well as the GGS groups have the congruence subgroup property [4, 10].

## 1.2 Current state of personal research

In the context of the profinite topology of branch groups, the right property to study is the so called *subgroup induction property* as I demonstrated with my coauthors. Indeed, we have

**Theorem 6** ([26, 19, 11]). *Let  $G$  be a finitely generated, branch group with the subgroup induction property. Then*

1.  $G$  is torsion and just infinite,
2. if  $G$  is a  $p$ -group, then all maximal subgroups of  $G$  are of finite index, and hence profinite-closed,

3. all weakly maximal subgroups of  $G$  are profinite-closed,
4. if  $G$  is self-similar and has the congruence subgroup property, it is LERF (i.e. all finitely generated subgroups are profinite-closed),
5.  $H \leq G$  is finitely generated if and only if there exists a block subgroup  $L \leq H$  such that the index  $[H : L]$  is finite.

Moreover, if  $G$  is self-similar, the item 5 is in fact equivalent to the subgroup induction property.

We will not give here a formal definition (see [26, 19, 11]) of the subgroup induction property or of block subgroups, but rather focus on some of its consequences. First of all, heuristically, a block subgroup of  $G$  is a subgroup which is isomorphic to a finite product of finite index subgroups of  $G$ , some of them embedded diagonally; Figures 2 and 1 might help to represent it. In particular, the description of a block subgroup use only a finite number of vertices of  $T$ .

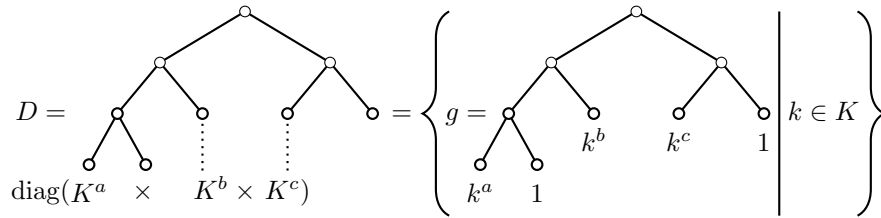


Figure 1: A diagonal subgroup of the first Grigorchuk group  $\mathcal{G}$ .  $K$  is a finite (16) index subgroup of  $\mathcal{G}$ . The diagonal subgroup consists in elements  $g \in \text{Aut}(T)$  that fix the leaves of the tree and act as  $k^a$  on the leftmost leaf, as the identity on the next leaf, etc.

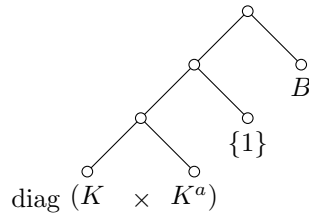


Figure 2: A block subgroup of the first Grigorchuk group  $\mathcal{G}$ .  $B$  and  $K$  are finite index (8 and 16) subgroups of  $\mathcal{G}$ . Here  $B$  under the rightmost vertex means that we consider elements of  $\text{Aut}(T)$  that fix this vertex  $v$  and act as an element of  $B$  under the subtree rooted at  $v$ . The final subgroup is the product of this copy of  $B$  and of the subgroup  $\text{diag}(K \times K^a)$ .

The most striking consequences of the subgroup induction property are described in Theorem 6, but there are not the only ones. For example, we have

**Theorem 7** ([11]). *Let  $G \leq \text{Aut}(T_d)$  be a self-replicating branch group where  $T_d$  is a  $d$ -regular tree. Suppose that  $G$  has the subgroup induction property and let  $H$  be a finitely generated subgroup of  $G$ . Then  $H$  is commensurable with one of  $\{1\}, G, G^2, \dots, G^{d-1}$ .*

*If moreover  $G$  is strongly self-replicating, has the congruence subgroup property and is a  $p$ -group, then all maximal subgroups of  $H$  are of finite index.*

The article [11] contains other consequences of the subgroup induction property, while [19] give several equivalent characterizations of it, some of them being better suited for applications than the original definition from [20]. We also have the following lemma underline the importance of the congruence subgroup property.

**Lemma 8** ([26]). *Let  $G \leq \text{Aut}(T)$  be a branch group with the congruence subgroup property. Then every subgroup with a block subgroup is closed for the profinite topology.*

Finally, the subgroup induction property has consequences on the cohomology of the group, as well as on its Cantor-Bendixon rank, see [12, 39, 11].

Since the subgroup induction property is a powerful tool, it is important to be able to exhibit groups that satisfy it. Until recently, only two groups were known to possess the subgroup induction property: the first Grigorchuk group [20] and the Gupta-Sidki 3-group [13]. With D. Francoeur we recently showed that all groups in some well-studied infinite family of branch groups have the desired property.

**Theorem 9** ([11]). *Let  $p$  be a prime and  $G \leq \text{Aut}(T_p)$  be a torsion GGS group. Then  $G$  has the subgroup induction property.*

Finally, in [26] I completely described the structure of weakly maximal subgroups of  $\mathcal{G}$  and torsion GGS groups, completing works from [6] and hence answering a question of Grigorchuk.

### 1.3 Detailed research plan

Since  $\mathcal{G}$  has the subgroup induction property, every finitely generated subgroup  $H$  contains a finite index block subgroup  $L$  which is builded using finite index subgroups  $L_i$  of  $\mathcal{G}$ . While this property is useful (for examples, it implies that  $\mathcal{G}$  is LERF), in practice it is complicated to deal with arbitrary finite index subgroups  $L_i$  of  $\mathcal{G}$ . We hence plan to replace the finite index subgroups appearing in the definition of a block subgroup by a unique subgroup  $K$  (of index 16) of  $\mathcal{G}$ . That is, we plan to prove the following

**Conjecture 10.** *Let  $H$  be a finitely generated subgroup of  $\mathcal{G}$ . Then there exists a block subgroup  $L \leq H$  that have only blocks over  $K$  and such that  $[H : L]$  is finite.*

This is an ongoing project with D. Francoeur, R. Grigorchuk, T. Nagibeda and M. Nocce. The general structure of the proof of Conjecture 10 is already pretty clear. The main idea is to show that if  $H$  is a finite index subgroup of  $\mathcal{G}$  that is isomorphic to  $K^n$ , then  $H$  is already “geometrically”  $K^n$ : that is  $H$  is a block subgroup with  $n$  blocks over  $K$ . In order to prove that we will use a rigidity result of Rubin [38] and the structure of the abstract group  $\text{Aut}(K)$ . We conjecture that  $\text{Aut}(K)$  is the normalizer  $N_{\text{Aut}(T_2)}(K)$  of  $K$  in the full group of automorphisms of the binary rooted tree  $T_2$ . A similar result is already known for  $\mathcal{G}$  itself:  $\text{Aut}(\mathcal{G}) = N_{\text{Aut}(T_2)}(\mathcal{G})$  [16] and preliminary results show that we should be able to obtain the same equality for  $K$ .

Another part of this project is to provide an algorithm that given  $\{g_1, \dots, g_n\}$  will return the block structure for  $\langle g_1, \dots, g_n \rangle$ . The main idea here is to rewrite in the context of  $\mathcal{G}$  the abstract proof of [19] (which works for all non-virtually abelian just infinite groups) in a more effective way in order to obtain the algorithm. We then plan to give another algorithm that given  $\{g_1, \dots, g_n\}$  and  $\{h_1, \dots, h_m\}$  will decide if  $\langle g_1, \dots, g_n \rangle \cap \langle h_1, \dots, h_m \rangle$  is finitely generated or not. Both algorithms require a good understanding of  $\text{Aut}(K)$ .

Once Conjecture 10 is proved and  $N_{\text{Aut}(T_2)}(K)$  is fully understood, I plan to give a positive answer to Question 4. Indeed, one advantage of block subgroups is that they are easier to deal with than general finitely generated subgroups. One easily sees that the main technical obstacle to understanding the product of two block subgroups is that diagonal blocks are defined using arbitrary automorphisms of the finite index subgroups of  $\mathcal{G}$ . But this is not a real problem if we know that only a specific subgroup  $K$  is involved, and we have a good description of  $\text{Aut}(K)$ .

As for Question 3, one must remind that any group can always be written as an increasing union of finitely generated subgroups. So let  $H$  be any subgroup of  $\mathcal{G}$ . Then  $H$  is the increasing union of finitely generated subgroups  $H_i$ . For each of these  $H_i$ , we have a block subgroup  $L_i$  and one can look at  $L := \bigcup_i L_i$ . The hope here is that  $L$  would be what can be called a *generalized block subgroup*, that is similar to a block subgroup but on infinitely many vertices of  $T$ . It follows from the method developed in [26] that such a subgroup is profinite-closed. It then remains to show that  $L$  is a finite index subgroup of  $H$ , or at least the fact that  $L$  is closed implies that  $H$  is also closed. If this approach shows to be unsuccessful, one can tackle the problem from another angle: try to directly prove that any subgroup of  $\mathcal{G}$  has a finite index subgroup which is a generalized block subgroup. The idea here is to proceed to do it in two steps. Firstly, we would need to show that

$\mathcal{G}$  satisfy some strong form of the subgroup induction property: namely that if  $H$  is a subgroup of  $\mathcal{G}$ , there exists a collection  $V$  (not necessarily finite) of pairwise incomparable vertices of  $T$  such that for any  $v$  in  $V$  the projection of  $H$  onto  $\text{Stab}_{\mathcal{G}}(v)$  has finite index and such that  $[H : \text{Stab}_H(V)]$  is finite, where  $\text{Stab}_H(V)$  is the pointwise stabilizer of  $V$ . If  $V$  is finite and  $H$  finitely generated, then this is the subgroup induction property. In a second time, we would need to adapt the argument from [19] to show that this property implies that  $H$  has a finite index subgroup which is a generalized block subgroup. Observe that even if  $\mathcal{G}$  is not ERF, the above strategy might be useful to characterize which subgroups of  $\mathcal{G}$  are profinite-closed.

As we have seen in the last section, the congruence subgroup property (the fact that the profinite topology on  $\mathcal{G}$  coincide with the topology induced by the natural topology of  $\text{Aut}(T)$ ) is an important property in our context. The following question merits to be studied more deeply.

**Question 11.** *Does the subgroup induction property implies the congruence subgroup property?*

While Theorem 9 give infinitely many examples of groups with the subgroup induction property, it gives us only  $p$ -groups (with  $p$  prime); and for a given  $p$ , only finitely many of them. One can hence ask

**Question 12.** *Given a prime number  $p$ , are there infinitely many numbers of  $p$ -groups with the induction subgroup property?*

*Does there exists a group with the subgroup induction property which is not a  $p$ -group for some prime  $p$ ?*

*Are there uncountably many groups with the induction subgroup property?*

A first step to answer these question is to look at other well-studied family of branch groups. One might cite GGS groups on the  $d$ -regular tree for  $d$  not prime, as well as the Šunić groups. This is a project in collaboration with A. Thillaisundaram and D. Francoeur.

Finally, one can wonder about the possibility to generalize the above results and questions to groups that are not necessarily branch. Indeed, if  $G \leq \text{Aut}(T)$  is branch, then the action of  $G$  on the boundary  $\partial T$  of the tree is what is called *micro-supported*. Many results first obtained for branch groups were later extended to groups with micro-supported actions. On one hand, we gave in [19] several equivalent definition of the subgroup induction property, one of which making sense in the more general context of micro-supported actions. On the other hand, some of the proofs of [11] also work in this generality. Nevertheless, for now all the known-examples of groups with the subgroup induction property are branch and it is unclear if there exist examples of non-branch groups with the subgroup induction property.

## 1.4 Schedule and milestones

A first article solving Conjecture 10, in collaboration with D. Francoeur, R. Grigorchuk, T. Nagnibeda and M. Nocce, should be published during the first year of the project. A second article resolving Question 4 should soon follow. This second article might be a good project for a PhD student.

The solution of Question 3 seems harder and may take more time. It will probably involve co-authors, and will be a good opportunity to collaborate with a postdoc or a PhD student.

Questions 11 and 12 as well as the generalization of the above results to groups with micro-supported action are relatively independent of the other parts. They can hence be attacked in parallel and I hope to obtain results in the first two years of the project. The generalization to groups with micro-supported actions will be greatly helped by the choice of a postdoc with knowledge on the subject, or with collaboration with specialist of the subject, as for example A. Leboudec or N. Matte Bon.

If some of these problems remain open, the organization of the conference in les Diablerets in 2026 will provide a good opportunity to discuss them with other specialists.

Altogether, the problems related to this part of the project should generate three to four articles.

## 1.5 Relevance and impact

This project lies at the intersection of branch groups and of the profinite topology, both subjects that have been deeply investigated these last years.

From a topological point of view, it would provide new examples of ERF groups and of groups answering positively Question 4. This is interesting as  $\mathcal{G}$  is far away to look like the only known examples of such groups (free groups and some of their generalization). Indeed,  $\mathcal{G}$  is torsion, amenable, has intermediate growth and so on. This would hence open a new direction (branch groups, groups with micro-supported actions) in which might want to search for ERF groups. These new examples would serve as test cases and help to better understand the possible restrictions on ERF groups.

From a branch groups point of view, this would allow to better understand the lattice of subgroup of  $\mathcal{G}$ . Indeed,  $\mathcal{G}$  and some other branch groups possess rather unusual properties and people try to understand these groups as well as possible. In this context the lattice of subgroups play a particularly important role, as one can define the “branchness” of a group using it. These continuing interest is behind the past and present exploration of some specific subgroups of  $\mathcal{G}$ , as for example: normal subgroups, finite index subgroups, maximal subgroups, weakly maximal subgroups and finitely generated subgroups.



## 2 Rigidity phenomenon in Cayley graphs

### 2.1 Current state of research in the field

Classically, a representation of a group is defined as an homomorphism  $G \rightarrow \text{GL}(V) = \text{Aut}(V)$  where  $V$  is a vector space, generally over  $\mathbf{C}$ . People have tried to generalize this to homomorphisms  $G \rightarrow \text{Aut}(X)$  where  $X$  is some “nice geometric space”. In the context of graph theory, one generally ask for  $G \cong \text{Aut}(X)$  as there is too much freedom for  $\text{Aut}(X)$ . This leads us to the question: What are the finitely generated groups  $G$  such that there exists a connected locally finite graph  $X$  with  $G \cong \text{Aut}(X)$ . Groot (1959) and Sabidussi (1960) have proved all finitely generated groups can be represented in this way. But we can still ask the same question, but restricted to graphs with more structure.

**Question 13.** *What are the finitely generated groups  $G$  such that there exists a connected locally finite graph  $X$  with  $G \cong \text{Aut}(X)$  and such that  $\text{Aut}(X)$  acts both freely and transitively on the vertices of  $X$ ?*

By a result of Sabidussi, this implies that  $X$  is a Cayley graph of  $G$ , of which we will recall the definition.

Let  $G$  be a group and  $S$  a symmetric generating set. One can associate to the couple  $(G, S)$  its *Cayley graph*  $\text{Cay}(G; S)$ . The vertices of  $\text{Cay}(G; S)$  are elements of  $G$  and there is an edge between  $g$  and  $h$  if and only if  $g^{-1}h$  is in  $S$ . This well-known construction allow to see  $G^1$  as a metric space. The group  $G$  naturally acts on  $\text{Cay}(G; S)$  by left multiplication, which gives us an embedding of  $G$  into the group  $\text{Aut}(\text{Cay}(G; S))$  of the automorphisms of  $\text{Cay}(G; S)$ . Question 13 is hence equivalent to

**Question 14.** *What are the finitely generated groups  $G$  such that there exists a finite generating set  $S$  for which the only automorphisms of  $\text{Cay}(G; S)$  are the translation by elements of  $G$ ?*

If  $\text{Cay}(G; S)$  is such a Cayley graph, we call it a *graphical rigid representation* (GRR) of  $G$ . A group  $G$  is *rigid* if it has at least one GRR. A simple verification shows that abelian groups (of exponent at least 3) are never rigid. Indeed, the map  $g \mapsto g^{-1}$  is always an element of  $\text{Aut}(\text{Cay}(G; S)) \setminus G$ . One can also show that the generalized dicyclic groups are not rigid. Finally, among groups of order at most 32, they are exactly 13 exceptional groups that are neither rigid, nor abelian nor dicyclic generalized. This and previous results lead Watkins to conjecture the following in 1976

**Conjecture 15** ([46]). *Apart the above counter-examples, every group is rigid.*

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<sup>1</sup>Or more precisely the couple  $(G; S)$ . But given two finite generating sets  $S$  and  $T$ , the graphs  $\text{Cay}(G; S)$  and  $\text{Cay}(G; T)$  are quasi-isometric.

With the combined effort of many mathematicians (notably Imrich, Watkins, Nowitz, Hetzel and Godsil) during the years 1969-1978, this conjecture was solved positively for finite groups [23, 9, 43, 36, 44, 45, 24, 25, 22, 15]. These proofs use deeply the fact that the groups under consideration are finite (for example, it uses the Feit-Thompson theorem), and they do not admit straightforward generalizations to infinite groups. Let us also mention that Babai and Godsil showed [3] that if  $G$  is a nilpotent non-abelian group of odd order, asymptotically almost all Cayley graphs of  $G$  are GRRs.

On the other hand, Watkins showed [30] that a free product of at least 2 and at most countably many non-trivial groups has a GRR. Moreover, if the group in question is finitely generated, then the GRR in question is locally finite. Here the method used is to start with a free group and then consider quotients of it.

Finally, Babai solved the directed version of this problem in a series of two papers [1, 2], and this without any assumptions on the cardinality of the groups under questions.

Recent developments in the subject were made in at least two distinct directions. The first one was the study of the Cayley graphs of  $G$  when  $G$  is either a finite abelian group of exponent greater than 2, [7], or a finite generalized dicyclic group, [34]. In both cases, the minimal index  $[\text{Aut}(\text{Cay}(G; S)) : G]$  was computed. The other recent development is the study of graphical Frobenius representations of a group  $G$ . They are graphs  $(V, E)$  such that  $G \cong \text{Aut}(V, E)$ , the group  $\text{Aut}(V, E)$  acts transitively on  $V$ , there exists some non-trivial elements in  $\text{Aut}(V, E)$  that fixes a vertex and no non-trivial elements of  $\text{Aut}(V, E)$  fixes more than 1 vertex, see [8, 47, 40].

## 2.2 Current state of personal research

Together with M. de la Salle, we recently proved Conjecture 15 for finitely generated infinite groups [28, 27]. The main idea of [28] was to split the problem of finding a GRR into two smaller problems that we solved separately. We showed that a group  $G$  is rigid if and only if there exists a generating set such that  $\text{Cay}(G; S)$  has two properties that we called *orientation-rigidity* and *color-rigidity*. One important thing about orientation-rigidity is that it is a monotone property: if  $S \subseteq T$  and  $\text{Cay}(G; S)$  is orientation-rigid, then  $\text{Cay}(G; T)$  is also orientation-rigid. We proved

**Theorem 16** ([28]). *Let  $G$  be a group that is neither abelian of exponent greater than 2 nor generalized dicyclic and let  $S$  be a symmetric generating set. Then  $\text{Cay}(G; T)$  is orientation-rigid for  $T = (S \cup S^2 \cup S^3) \setminus \{1\}$ .*

Observe that this result holds for any group, not necessarily finitely generated, and that  $T$  is finite if  $S$  is finite.

We then proceed to prove

**Theorem 17** ([28]). *Let  $G$  be a finitely generated group with an element of infinite order. Let  $S$  be a finitely generated set. Then there exists a finite  $S \subset T$  such that  $\text{Cay}(G; T)$  is color-rigid.*

The main technical tool involved here is the counting of the number of triangles to which belong an edge of  $\text{Cay}(G; S)$  labeled by some  $s \in S$ .

By combining Theorems 16 and 17 one obtains a proof of Conjecture 15 for finitely generated groups with an element of infinite order.

The second article [27] was devoted to the proof of

**Theorem 18** ([27]). *Let  $G$  be a finitely generated group which is not virtually abelian. Let  $S$  be a finitely generated set. Then there exists a finite  $S \subset T$  such that  $\text{Cay}(G; T)$  is color-rigid.*

The proof of Theorem 18 is probabilistic and uses results of M. Tointon [42] about commuting probabilities in finitely generated groups.

Using that finitely generated groups are either virtually abelian or have an element of infinite order, we can conclude that Conjecture 15 is proven for finitely generated groups. Moreover, the results of [28, 27] can be viewed as a weak asymptotic statement. Indeed, if  $G$  is an infinite finitely generated rigid group, then for any generating set  $S$  our proof provides a generating set  $S \subseteq T$  witnessing the rigidity of  $G$  and with  $|T| \leq f(|S|)$  for some explicit function  $f$ .

Finally, in [29] we computed the minimal index  $[\text{Aut}(\text{Cay}(G; S)) : G]$  for finitely generated groups with no GRR.

### 2.3 Detailed research plan

The aim here is to prove Conjecture 15 for groups that are not finitely generated. To do that, it seems promising to combine results from [28] and [1]. Indeed, both Theorem 16 and the results of Babai on the directed analog of Conjecture 15 hold for general groups. For the general case, the idea is to replace *finite generating set* by “small generating set”. We then need to define precisely what we mean by “small” and to show

**Conjecture 19.** *Let  $G$  be an infinite group. Then  $G$  always possesses a small generating set  $S$  and for every such  $S$  there exists a small  $S \subset T$  such that  $\text{Cay}(G; T)$  is orientation-rigid.*

**Conjecture 20.** *Let  $G$  be an infinite group and  $T$  be a small generating set. Then there always exists a (small) set  $T \subset V$  such that  $\text{Cay}(G; V)$  is color-rigid.*

While the general case seems hard, we are confident that the cases of countable groups is doable. In this case, we can take for the definition of “small”: every edge of  $\text{Cay}(G; S)$  belongs to at most finitely many triangles. In this context, I was able to prove:

**Lemma 21.** *Every countable group  $G$  admits a generating set  $S$  such that every edge of  $\text{Cay}(G; S)$  belongs to at most finitely many triangles.*

However, this is not enough to prove Conjecture 19 and we should prove the existence of an  $S$  satisfying more combinatorial conditions (as for example:  $xyz \neq 1$  if  $\{x, x^{-1}\}$ ,  $\{y, y^{-1}\}$  and  $\{z, z^{-1}\}$  are pairwise distincts). One can adapt ideas of Babai [1] to show the existence of an  $S$  satisfying a subset of the desired conditions, but the desired result require more work. However, even if the combinatorics involved are more complex, this seems feasible.

For Conjecture 20, we plan to do a dichotomy depending if  $G$  is virtually abelian or not. Observe that a countable group might be both torsion and not-virtually abelian!

If  $G$  is a countable virtually abelian group, the idea is to do two steps. Firstly, we want to show that if  $H$  is a countable abelian group, then there exists a “small” generating set  $S$  such that  $[\text{Aut}(\text{Cay}(H; S)) : H] = 2$ ; a fact that we already proved to be true for infinite finitely generated abelian groups [29]. If the combinatoric is more complex in the countable case than in the finitely generated case, one can still hope for a not too hard proof of this fact, at least if  $H$  has an element of infinite order. More precisely, it is well-known that a countable abelian group either has an element of infinite order or is a direct sum  $\bigoplus_p A_p$  where the  $A_p$  are abelian  $p$ -groups,  $p$  prime. If  $H$  has an element of infinite order, or if all the  $A_p$  have finite exponent, then the methods of [29] should generalize. However, the case where at least one  $A_p$  has elements of unbounded order seems more complicated. Once this step done, we will look at virtually abelian groups  $G$ . In this case, there exists a finite subset  $T \subset G$  and an abelian subgroup  $H$  such that  $G = \langle H, T \rangle$ . By combining carefully the results on countable abelian groups and on finitely generated groups, one might hope for a proof of Conjecture 15 for virtually countable groups.

We now look at the non-virtually abelian case. The idea is to generalize results of [27] which uses random walks and some ideas of Tointon [42] about commuting probabilities. Results of [42] are written for random walk  $(X_n)$  of law  $(\mu^{*n})$  where  $\mu$  is a symmetric measure of finite generating support containing  $\{1\}$ . For the countable case, we plan to generalize Tointon’s result to the following context. Let  $S = \{s_1, s_1^{-1}, s_2, s_2^{-1}, \dots\}$  and let  $\mu_n$  be the uniform measure on  $\{1, s_1^{\pm 1}, s_2^{\pm 1}, \dots, s_n^{\pm 1}\}$ . We then look at the sequence of random variables  $(X_n)$  of law  $(\mu_n^{*n})$ . If  $S$  is finite, then for  $n \geq |S|$  this is the classic random walk. If  $S$  is infinite, this is not anymore a random walk, but this looks like sufficiently like it so it is reasonable to hope to generalize results of [42] and [27] to this context.

Another possibility to solve the non-virtually abelian case is to look at the following conjecture and to its generalization to the countable case.

**Conjecture 22** ([27]). *Let  $G$  be a finitely generated non-virtually abelian group. Let  $\mu$  be a symmetric probability measure with finite and generating*

support containing the identity. Let  $(g_n)$  be a realisation of the random walk on  $G$  according to  $\mu$ . Then

$$\forall a \in G, \lim_n \mathbf{P}(g_n^2 = a) = 0.$$

Indeed, Theorem 18 follows relatively easily from this conjecture and the countable case should not be so different. In fact, the whole point of [27] was to prove a weaker version of Conjecture 22 and then to still derive Theorem 18 from it, but with a rather complicated proof. Observe that a quantitative version of Conjecture 22 is known to hold for finite groups [31, 32]. While the proofs of [31, 32] use character theory and cannot be pushed to infinite groups, a recent post from Terence Tao on *mathoverflow* [41] gives a character-free proof of the result of Mann for finite groups. With M. de la Salle we plan to generalize Tao's technique to finitely generated groups (and then to countable groups), using random walks techniques as initiated in [42] and used in [27].

## 2.4 Schedule and milestones

The part about virtually abelian groups and the one about non-virtually abelian groups are relatively independent one from the other, and one can start to work on one or on the other, as well as doing both simultaneously. The exact schedule can hence vary. Moreover, the following schedule assume that most of the first two years will be spend on branch groups, while in reality both projects would advance in parallel. Nevertheless, here is one possibility. I plan to prove Conjecture 19 for countable in the third year of the project. This could be done in collaboration with a PhD student or let completely to their. The proof of Conjecture 15 for virtually abelian groups should follow in the next 6 to 12 months.

The year 2027 will be devoted to the generalization of the results of Tointon to countable groups. The help of a postdoc with good knowledge of probability theory on groups will be greatly appreciated. Once this done, we will finish the proof of Conjecture 15 for non-virtually abelian groups. In parallel, we will look at Conjecture 22, first giving a proof of it for finitely generated groups, and then extending it to countable groups.

Finally, we will investigate which of the aforementioned results extend beyond the realm of countable groups. This last part is more difficult than the rest of the project and will probably require some deep combinatorics on sets. It is also more prone to failure, but not totally unachievable.

This project should produces around 4 articles.

## 2.5 Relevance and impact

On one hand, the original question of which groups can be represented as an automorphism groups of a suitably regular graph is itself of great interest and has attired the attention of many mathematicians. It took 10 years to

solve this question for finite groups, and then a waiting of 40 years before having the solution for finitely generated groups. I strongly believe that we have now the right tools to attack it and to hope for a full resolution of it in the next years. Observe that the (easier) analog for directed graphs is known to be true for any groups since 1980.

On the other hand, the technics used to attack this problem turned out to be powerful and versatile. Indeed, not only the articles [28, 27] solved the original problem for finitely generated groups, the same proofs gave essentially for free the results for the directed variant (the generating set is  $S$  not necessary symmetric and  $\vec{\text{Cay}}(G; S)$  is a digraph) and the oriented variant (for any  $x \in S$ , the element  $x^{-1}$  does not belong to  $S$ ) for finitely generated groups. The directed variant was already solved by Babai [1], but with the caveat of always providing an infinite generating set, while [28, 27] gave a finite generating set. For the oriented variant, this solved a conjecture raised by Babai in [28, 27]. Observe that the proof of the oriented variant for finite groups was only obtained in 2018 and use the classification of finite simple groups [33]. While in the directed case there is nothing more to prove for groups that are not finitely generated, the oriented version is still open. Our plan to prove Watkin’s conjecture for countable groups will probably give in the same time a proof for the oriented version of the conjecture. Other consequences of the results of [28, 27] include rigidity phenomenon for coverings  $\text{Cay}(G; S) \rightarrow X$  where is either transitive or a Schreier graph of  $G$ , see [28] for more details. Once again, a solution to Conjecture 15 for countable groups is likely to come with generalizations of the aforementioned rigidity results.

Finally, the technics of [42] and [27] that we plan to generalize further to countable groups are themselves of interest. Indeed, they allow to push to infinite groups some results known to be true for finite groups, as “if half of the elements of a finite group  $G$  are of order 2, then  $G$  is abelian” that for finitely generated groups became “if  $\mathbf{P}(g^2 = 1) > \frac{\sqrt{5}-1}{2}$ , then  $G$  is virtually abelian (for a suitable probability measure  $\mathbf{P}$ )”. It will be nice to obtain similar results for countable groups or even for arbitrary groups.

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