Closed subgroups and Cayley graphs of infinite groups

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My research interests lie in geometric and combinatorial group theory, as well as in symbolic dynamic. I am particularly interested in graphs of dynamical origin, groups acting on graphs (for examples on rooted trees) and their closed subgroups and on graphs associated with groups actions (Cayley and Schreier graphs) and rigidity phenomenon associated to them. Generally speaking, graphs (either related to applied questions, or viewed from a more abstract point of view), play an important role in my works and projects.

Below I will develop two specific points. These projects are independent and can be done one after the other or in parallel, in which case the schedule and milestones part should be adapted to take this in account.

1 Closed subgroups of branch groups

1.1 Current state of research in the field

This project is at the intersections of three important themes in geometric group theory: branch groups, self-similar groups and finally the profinite topology and the (locally) extended residually finite groups (i.e. (L)ERF groups).

A group is *just infinite* if it is infinite and all its proper quotients are finite. For an infinite group, this is a natural relax of the simplicity condition: we allow G to have non-trivial quotients, but only if they are finite, and hence in some sense trivial from the geometric group theory point of view. This condition is both natural from the large-scale geometry point of view, and useful in practice as every finitely generated infinite group admits a just infinite quotient. A major discovery of Wilson in the 70' (later refined by Grigorchuk) is that the class of just infinite groups splits into three subclasses which roughly are infinite simple groups, herediteraly just infinite groups (every finite index subgroup of G is also just infinite) and just infinite branch groups. Groups in the last category are examples of groups acting faithfully on a rooted tree and are characterised by a rich structures of subgroups, despite being just infinite. One striking example of such a group is the first Grigorchuk group $\mathcal{G}$, which was the first example of a group of intermediate growth, and hence also the first example of an amenable but not elementary amenable group [18, 19]. On the other hand, self-similar groups naturally appear in dynamics [39] and are examples of automata groups. Although quite different, these two classes of groups have a large intersection, and many self-similar groups are also branch. In the class of finitely generated branch self-similar groups, there are torsion groups and torsion free groups;

groups of intermediate growth and groups of exponential growth; amenable and nonamenable groups. Self-similar branch groups are a great source of examples and counter-examples in group theory and the study of these groups and their subgroup structure as attired a lot of attention among geometric group theorist these last years.

Another interesting subject in geometric group theory is the profinite topology. That is, the topology on G generated by the finite index subgroups (and their left cosets). A group G is *residually finite* if the trivial subgroup is closed in the profinite topology. Residually finite groups are a natural generalisation of finite groups, and hence are an interesting object of studies. Here are some of their many characterisations.

Fact 1. *For a group G, the following are equivalent*

1. *G is residually finite,*
2. *the profinite topology on G is Hausdorf,*
3. *the intersection of normal subgroups of finite index of G is trivial,*
4. *For every $g \neq 1$, there exists a finite quotient $\pi\colon G \twoheadrightarrow H$ such that $\pi(g) \neq 1$,*
5. *G embeds in its profinite completion,*
6. *(if G is finitely generated) G acts faithfully on a rooted tree.*

By definition, for every group G, finite index subgroups are profinite-closed (i.e. closed in the profinite topology), and if G is residually finite, then finite subgroups are also profinite closed. Given a group G, one can then ask the following

Question 2. *What are the profinite closed subgroups of G?*

If every subgroup of G is profinite closed, we will say that G is *extended residually finite* (ERF), while G is called *locally extended residually finite* (LERF) if all its finitely generated subgroups are profinite-closed.

While the class of residually finite groups contains a lot of well-known examples and has many distinct characterisations, the classes of LERF and ERF subgroups are less understood. As examples of ERF groups we have finite groups, finitely generated abelian groups, and virtually polycyclic groups [36]. The class of LERF groups contains all ERF groups, as well as finitely generated free groups, surface groups and more generally limits groups [52]. Another interesting properties of finitely generated free groups is that a product $\prod_{i=1}^n H_i$ of finitely many finitely generated subgroups is closed. This remarkable property is known (for finitely generated groups) only for free groups and some straightforward generalisations.

Since branch groups are by definition subgroups of $\mathrm{Aut}(T)$, where T is a rooted tree, they are residually finite. One can hence ask the following.

Question 3. *Let $\mathcal{G}$ be the first Grigorchuk group. Is it possible to describe all its profinite-closed subgroups?*

Question 4 ([23])**.** *Let H_1 to H_n be finitely generated subgroups of $\mathcal{G}$. Is the subset $\prod_{i=1}^n H_i$ profinite closed?*

And more generally

Question 5. *Which branch groups are LERF?*

Some partial results to the above question are already known. For example, all maximal subgroups of $\mathcal{G}$ are of finite index [41] and hence profinite closed. By [22], this implies that the weakly maximal subgroups of $\mathcal{G}$ (subgroups maximal among infinite index subgroups) are also closed. Finally, it follows from [22] that $\mathcal{G}$ is LERF. It is also known that $\mathcal{G}$ cannot be ERF, as it contains subgroups isomorphic to $\bigoplus_{n\geq 1} C_{2^n}$ which is not ERF.

Finally, an important property in the context of closed subgroups of $G \leq \mathrm{Aut}(T)$ is the so-called *congruence subgroup property* which asserts that the profinite topology on G coincide with the restriction to G of the natural topology of $\mathrm{Aut}(T)$ (i.e. the topology generated by the pointwise stabilisers of the levels of the tree). Indeed, if $G \leq \mathrm{Aut}(T)$ has the congruence subgroup property, then in order to understand the profinite topology of G we don't need to look at all finite index subgroups, but it is enough to look at the stabilisers of the levels and we can hence more geometric methods. Due to its importance, this property has been extensively investigated these last twenty years, see [6, 15] and the references therein. In particular, it is known that $\mathcal{G}$ as well as the GGS groups have the congruence subgroup property [5, 11].

1.2 Current state of personal research

In the context of the profinite topology of branch groups, the right property to study is the so called *subgroup induction property* as I demonstrated with my coauthors. Indeed, we have

Theorem 6 ([32, 20, 12])**.** *Let G be a finitely generated, branch group with the subgroup induction property. Then*

1. *G is torsion and just infinite,*
2. *if G is a p-group, then all maximal subgroups of G are of finite index, and hence profinite-closed,*
3. *all weakly maximal subgroups of G are profinite-closed,*
4. *if G is self-similar and has the congruence subgroup property, it is LERF (i.e. all finitely generated subgroups are profinite-closed),*
5. *$H \leq G$ is finitely generated if and only if there exists a* block subgroup *$L \leq H$ such that the index $[H : L]$ is finite.*

Moreover, if G is self-similar, the item 5 is in fact equivalent to the subgroup induction property.

We will not give here a formal definition (see [32, 20, 12]) of the subgroup induction property or of block subgroups, but rather focus on some of its consequences. First of all, heuristically, a block subgroup of G is a subgroup which is isomorphic to a finite product of finite index subgroups of G, some of them embedded diagonally; Figures 2 and 1 might helps to represent it. In particular, the description of a block subgroup use only a finite number of vertices of T.

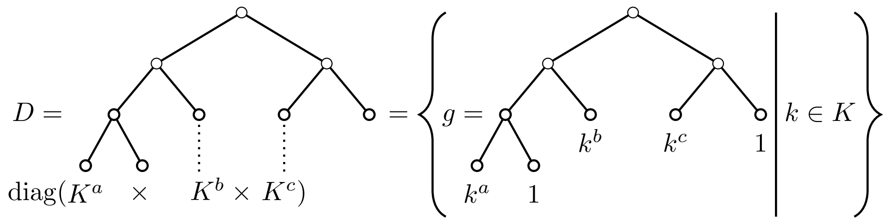

Figure 1: A diagonal subgroup of the first Grigorchuk group $\mathcal{G}$. K is a finite (16) index subgroup of $\mathcal{G}$. The diagonal subgroup consists in elements $g \in \text{Aut}(T)$ that fixe the leaves of the tree and act as k^a on the leftmost leaf, as the identity on the next leaf, etc.

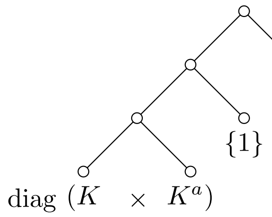

Figure 2: A block subgroup of the first Grigorchuk group $\mathcal{G}$. B and K are finite index (8 and 16) subgroups of de $\mathcal{G}$. Here B under the rightmost vertex means that we consider elements of $\text{Aut}(T)$ that fixe this vertex v and act as an element of B under the subtree rooted at v. The final subgroup is the product of this copy of B and of the subgroup $\text{diag}(K \times K^a)$.

The most striking consequences of the subgroup induction property are described in Theorem 6, but there are not the only ones. For example, we have

Theorem 7 ([12])**.** *Let $G \leq \text{Aut}(T_d)$ be a self-replicating branch group where T_d is a d-regular tree. Suppose that G has the subgroup induction property and let H be a finitely generated subgroup of G. Then H is commensurable with one of $\{1\}$, G, G^2,..., G^{d-1}.*

If moreover G is strongly self-replicating, has the congruence subgroup property and is a p-group, then all maximal subgroups of H are of finite index.

The article [12] contains other consequences of the subgroup induction property, while [20] give several equivalent characterisations of it, some of them being better suited for applications than the original definition from [22]. We also have the following lemma underline the importance of the congruence subgroup property.

Lemma 8 ([32])**.** *Let $G \leq \text{Aut}(T)$ be a branch group with the congruence subgroup property. Then every subgroup with a block subgroup is closed for the profinite topology.*

Finally, the subgroup induction property has consequences on the cohomology of the group, has well as on its Cantor-Bendixon rank, see [13, 46, 12].

Since the subgroup induction property is a powerful tool, it is important to be able to exhibit groups that satisfy it. Until recently, only two groups were known to possess the subgroup induction property: the first Grigorchuk group [22] and the Gupta-Sidki 3-group [14]. With D. Francoeur we recently showed that all groups in some well-studied infinite family of branch groups have the desired property.

Theorem 9 ([12])**.** *Let p be a prime and $G \leq Aut(T_p)$ be a torsion GGS group. Then G has the subgroup induction property.*

Finally, in [32] I completely described the structure of weakly maximal subgroups of $\mathcal{G}$ and torsion GGS groups, completing works from [7] and hence answering a question of Grigorchuk.

1.3 Detailed research plan

Since $\mathcal{G}$ has the subgroup induction property, every finitely generated subgroup H contains a finite index block subgroup L which is builded using finite index subgroups L_i of $\mathcal{G}$. While this property is useful (for examples, it implies that $\mathcal{G}$ is LERF), in practice it is complicated to deal with arbitrary finite index subgroups L_i of $\mathcal{G}$. We hence plan to replace the finite index subgroups appearing in the definition of a block subgroup by a unique subgroup K (of index 16) of $\mathcal{G}$. That is, we plan to prove the following

Conjecture 10. *Let H be a finitely generated subgroup of $\mathcal{G}$. Then there exists a block subgroup $L \leq H$ that have only blocks over K and such that $[H : L]$ is finite.*

This is on ongoing project with D. Francoeur, R. Grigorchuk and T. Nagnibeda. The general structure of the proof of Conjecture 10 is already pretty clear. The main idea is to show that if H is a finite index subgroup of $\mathcal{G}$ that is isomorphic to K^n, then H is already "geometrically" K^n: that is H is a block subgroup with n blocks over K. In order to prove that we will use a rigidity result of Rubin [45] and the structure of the abstract group $\mathrm{Aut}(K)$. We conjecture that $\mathrm{Aut}(K)$ is the normaliser $N_{\mathrm{Aut}(T_2)}(K)$ of K in the full group of automorphisms of the binary rooted tree T_2. A similar result is already known for $\mathcal{G}$ itself: $\mathrm{Aut}(\mathcal{G}) = N_{\mathrm{Aut}(T_2)}(\mathcal{G})$ [17] and preliminary results show that we should be able to obtain the same equality for K.

Another part of this project is to provide an algorithm that given $\{g_1, \ldots, g_n\}$ will return the block structure for $\langle g_1, \ldots, g_n \rangle$. The main idea here is to rewrote in the context f $\mathcal{G}$ the abstract proof of [20] (which works for all non-virtually abelian just infinite groups) in a more effective way in order to obtain the algorithm. We then plan to give another algorithm that given $\{g_1, \ldots, g_n\}$ and $\{h_1, \ldots, h_m\}$ will decide if $\langle g_1, \ldots, g_n \rangle \cap \langle h_1, \ldots, g_m \rangle$ is finitely generated or not. Both algorithm require a good understanding of $\mathrm{Aut}(K)$.

Once Conjecture 10 proved and $N_{\mathrm{Aut}(T_2)}(K)$ fully understood, I plan to give a positive answer to Question 4. Indeed, one advantage of the block subgroups is that they are easier to deal with than general finitely generated subgroups. One easily see that the main technical obstacle to understand the product of two block subgroups is that diagonal blocks are defined using arbitrary automorphisms of the finite index subgroups of $\mathcal{G}$. But this is not a real problem if we know that only a specific subgroup K is involved, and we have a good description of $\mathrm{Aut}(K)$.

As for Question 3, one must remind that any group can always be written as an increasing union of finitely generated subgroups. So let H be any subgroup of $\mathcal{G}$. Then H is the increasing union of finitely generated subgroups H_i. For each of these H_i, we have a block subgroup L_i and one can look at $L \coloneqq \bigcup_i L_i$. The hope here is that L would be what can called a *generalised block subgroup*, that is similar to a block subgroup but on infinitely many vertices of T. It follows from the method developed in [32] that such a subgroup is profinite-closed. It will then remain to show that L as finite index in H, or at least the fact that L closed implies that H is also closed. As $\mathcal{G}$ is not ERF, this approach will sometimes fails. In fact, it is easy to see that for the copies of $\bigoplus_{n\geq 1} C_{2^n}$ contained in $\mathcal{G}$, all the L_i will be trivial and so will be L. More than a defect of the method, this is an hint for which subgroups are not closed for "obvious" reasons, as for examples increasing unions of finite groups. One can hope to use the above method to provide a characterisation of profinite-closed subgroups of $\mathcal{G}$.

As we have seen in the last section, the congruence subgroup property (the fact that the profinite topology on $\mathcal{G}$ coincide with the topology induced by the natural topology of $\mathrm{Aut}(T)$) is an important property in our context. The following question merits to be studied more deeply.

Question 11. *Does the subgroup induction property implies the congruence subgroup property?*

While Theorem 9 give infinitely many examples of groups with the subgroup induction property, it gives us only p-groups (with p prime); and for a given p, only finitely many of them. One can hence ask

Question 12. *Given a prime number p, are they infinitely many numbers of p-groups with the induction subgroup property?*

Does there exists a group with the subgroup induction property which is not a p-group for some prime p?

Are there uncountably many groups with the induction subgroup property?

A first step to answer these question is to look at other well-studied family of branch groups. One might cite GGS groups on the d-regular tree for d not prime, as well as the Šunić groups. This is a project in collaboration with A. Thillaisundaram and D. Francoeur.

Finally, one can wonder about the possibility to generalise the above results and questions to groups that are not necessarily branch. Indeed, if $G \leq \mathrm{Aut}(T)$ is branch, then the action of G on the boundary ∂T of the tree is what is called *micro-supported*. Many results first obtained for branch groups where later extended to groups with micro-supported actions. On one hand, we gave in [20] several equivalent definition of the subgroup induction property, one of which making sense in the more general context of micro-supported actions. On the other hand, some of the proofs of [12] also work in this generality. Nevertheless, for now all the known-examples of groups with the subgroup induction property are branch and it is unclear if there exist examples of non-branch groups with the subgroup induction property.

1.4 Relevance and impact

This project lies at the intersection of branch groups and of the profinite topology, both subjects that have been deeply investigated these last years.

From a topological point of view, it would provide new examples of ERF groups and of groups answering positively Question 4. This is interesting as $\mathcal{G}$ is far away to look like the only known examples of such groups (free groups and some of their generailisation). Indeed, $\mathcal{G}$ is torsion, amenable, has intermediate growth and so on. This would hence open a new direction (branch groups, groups with micro-supported actions) in which might want to search for ERF groups. These new examples would serve as test cases and help to better understand the possible restrictions on ERF groups.

From a branch groups point of view, this would allow to better understand the lattice of subgroup of $\mathcal{G}$. Indeed, $\mathcal{G}$ and some other branch groups possess rather unusual properties and people try to understand these groups as well as possible. In this context the lattice of subgroups play a particularly important role, as one can define the "branchness" of a group using it. These continuing interest is behind the past and present exploration of some specific subgroups of $\mathcal{G}$, as for example: normal subgroups, finite index subgroups, maximal subgroups, weakly maximal subgroups and finitely generated subgroups.

2 Rigidity phenomenon in Cayley graphs

2.1 Current state of research in the field

Classically, a representation of a group is defined as an homomorphism $G \to \mathrm{GL}(V) = \mathrm{Aut}(V)$ where V is a vector space, generally over $\mathbf{C}$. People have tried to generalise this to homomorphisms $G \to \mathrm{Aut}(X)$ where X is some "nice geometric space". In the context of graph theory, one generally ask for $G \cong \mathrm{Aut}(X)$ as there is too much freedom for $\mathrm{Aut}(X)$. This leads us to the question: What are the finitely generated groups G such that there exists a connected locally finite graph X with $G \cong Aut(X)$. Frucht (1939), Groot (1959) and Sabidussi (1960) have proved all groups can be represented in this way. But we can still ask the same question, but restricted to graphs with more structure.

Question 13. *What are the finitely generated groups G such that there exists a connected locally finite graph X with $G \cong Aut(X)$ ans such that* $\mathrm{Aut}(X)$ *acts both freely and transitively on the vertices of X?*

By a result of Sabidussi, this implies that X is a Cayley graph of G, of which we will recall the definition.

Let G be a group and S a symmetric generating set. One can associate to the couple (G, S) its *Cayley graph* $\mathrm{Cay}(G; S)$. The vertices of $\mathrm{Cay}(G; S)$ are elements of G and there is an edge between g and h if and only if $g^{-1}h$ is in S. This well-known construction allow to see G[1] as a metric space. The group G naturally acts on $\mathrm{Cay}(G; S)$ by left multiplication, which gives us an embedding of G into the group $\mathrm{Aut}(\mathrm{Cay}(G; S))$ of the automorphisms of $\mathrm{Cay}(G; S)$. Question 13 is hence equivalent to

Question 14. *What are the finitely generated groups G such that there exists a finite generating set S for which the only automorphisms of* $\mathrm{Cay}(G; S)$ *are the translation by elements of G?*

[1] Or more precisely the couple $(G; S)$. But given two finite generating sets S and T, the graphs $\mathrm{Cay}(G; S)$ and $\mathrm{Cay}(G; T)$ are quasi-isometric.

If $\mathrm{Cay}(G;S)$ is such a Cayley graph, we call it a *graphical rigid representation* (GRR) of G. A group G is *rigid* if it has at least one GRR. A simple verification shows that abelian groups (of exponent at least 3) are never rigid. Indeed, the map $g \mapsto g^{-1}$ is always an element of $\mathrm{Aut}(\mathrm{Cay}(G;S)) \setminus G$. One can also show that the generalised dicyclic groups are not rigid. Finally, among groups of order at most 32, they are exactly 13 exceptional groups that are neither rigid, nor abelian nor dicyclic generalised. This and previous results lead Watkins to conjecture the following in 1976

Conjecture 15 ([51])**.** *Appart the above counter-examples, every group is rigid.*

With the combined effort of many mathematicians (notably Imrich, Watkins, Nowitz, Hetzel and Godsil) during the years 1969-1978, this conjecture was solved positively for finite groups [26, 10, 48, 40, 49, 50, 27, 28, 24, 16]. These proofs use deeply the fact that the groups under consideration are finite (for example, it uses the Feit-Thompson theorem), and they do not admit straightforward generalisations to infinite groups. Let us also mention that Babai and Godsil showed [3] that if G is a nilpotent non-abelian group of odd order, asymptotically almost all Cayley graphs of G are GRRs.

On the other hand, Watkins showed [30] that a free product of at least 2 and at most countably many non-trivial groups has a GRR. Moreover, if the group in question is finitely generated, then the GRR in question is locally finite. Here the method used is to start with a free group and then consider quotients of it.

Finally, Babai solved the directed version of this problem in a series of two papers [1, 2], and this without any assumptions on the cardinality of the groups under questions.

Recent developments in the subject were made in at least three distinct directions for finite groups. The first one was the study of the Cayley graphs of G when G is either a finite abelian group of exponent greater than 2, [8], or a finite generalised dicyclic group, [38]. In both cases, the minimal index $[\mathrm{Aut}(\mathrm{Cay}(G;S)) : G]$ was computed. The second other recent development is the study of some variations of this problems. One can for example ask that the GRR has small degree, that the Automorphism group acts freely but with n-orbits on the vertices (n fixed), ... Finally, Xia and Zheng proved recently [53] that for finite groups, asymptoticaly a randomly chosen generating set will give raise to a GRR.

2.2 Current state of personal research

Together with M. de la Salle, we recently proved Conjecture 15 for finitely generated infinite groups [34, 33]. The main idea of [34] was to split the problem of finding a GRR into two smaller problems that we solved separately. We showed that a group G is rigid if and only if there exists a generating set such that $\mathrm{Cay}(G;S)$ has two properties that we called *orientation-rigidity* and *color-rigidity*. One important thing about orientation-rigidity is that it is a monotonous property: if $S \subseteq T$ and $\mathrm{Cay}(G;S)$ is orientation-rigid, then $\mathrm{Cay}(G;T)$ is also orientation-rigid. We proved

Theorem 16 ([34])**.** *Let G be a group that is neither abelian of exponent greater than 2 nor generalised dicyclic and let S be a symmetric generating set. Then $\mathrm{Cay}(G;T)$ is orientation-rigid for $T = (S \cup S^2 \cup S^3) \setminus \{1\}$.*

Observe that this result holds for any group, not necessarily finitely generated, and that T is finite if S is finite.

We then proceed to prove

Theorem 17 ([34])**.** *Let G be a finitely generated group with an element of infinite order. Let S be a finitely generated set. Then there exists a finite $S \subset T$ such that $\mathrm{Cay}(G;T)$ is color-rigid.*

The maint technical tool involve here is the counting of the number of triangles to which belong an edge of $\mathrm{Cay}(G;S)$ labeled by some $s \in S$.

By combining Theorems 16 and 17 one obtain a proof of Conjecture 15 for finitely generated groups with an element of infinite order.

The second article [33] was devoted to the proof of

Theorem 18 ([33])**.** *Let G be a finitely generated group which is not virtually abelian. Let S be a finitely generated set. Then there exists a finite $S \subset T$ such that $\mathrm{Cay}(G;T)$ is color-rigid.*

The proof of Theorem 18 is probabilistic and use results of M. Tointon [47] about commuting probabilities in finitely generated groups.

Using that finitely generated groups are either virtually abelian or have an element of infinite order, we can conclude and Conjecture 15 is proven for finitely generated groups. Moreover, the results of [34, 33] can be viewed as a weak asymptotic statement. Indeed, if G is an infinite finitely generated rigid group, then for any generating set S our proof provide a generating set $S \subseteq T$ witnessing the rigidity of G and with $|T| \leq f(|S|)$ for some explicit function f.

Finally, in [35] we computed the minimal index $[\mathrm{Aut}(\mathrm{Cay}(G;S)) : G]$ for finitely generated groups with no GRR.

2.3 Detailed research plan

The aim here is to prove Conjecture 15 for groups that are not finitely generated. To do that, it seems promising to combine results from [34] and [1]. Indeed, both Theorem 16 and the results of Babai on the directed analog of Conjecture 15 hold for general groups. For the general case, the idea is to replace *finite generating set* by "small generating set". We then need to define precisely what we mean by "small" and to show

Conjecture 19. *Let G be an infinite group. Then G always possess a be a small generating set S and for every such S there exists a small $S \subset T$ such that $\mathrm{Cay}(G;T)$ is orientation-rigid.*

Conjecture 20. *Let G be an infinite group and T be a small generating set. Then there always exists a (small) set $T \subset V$ such that $\mathrm{Cay}(G;V)$ is color-rigid.*

While the general case seems hard, we are confident that the cases of countable groups is doable. In this case, we can take for the definition of "small": every edge of $\mathrm{Cay}(G;S)$ belongs to at most finitely many triangles. In this context, I was able to prove:

Lemma 21. *Every countable group G admits a generating set S such that every edge of $\mathrm{Cay}(G;S)$ belongs to at most finitely many triangles.*

However, this is not enough to prove Conjecture 19 and we should prove the existence of an S satisfying more combinatorial conditions (as for example: $xyz \neq 1$ if $\{x, x^{-1}\}$, $\{y, y^{-1}\}$ and $\{z, z^{-1}\}$ are pairwise distincts). One can adapt ideas of Babai [1] to show the existence of an S satisfying a subset of the desired conditions, but the desired result require more work. However, even if the combinatorics involved are more complex, this seems feasible.

For Conjecture 20, we plan to do a dichotomy depending if G is virtually abelian or not. Observe that a countable group might be both torsion and not-virtually abelian!

If G is a countable virtually abelian group, the idea is to do two steps. Firstly, we want to show that if H is a countable abelian group, then there exists a "small" generating set S such that $[\mathrm{Aut}(\mathrm{Cay}(H; S)) : H] = 2$; a fact that we already proved to be true for infinite finitely generated abelian groups [35]. If the combinatoric is more complex in the countable case than in the finitely generated case, one can still hope for a not to hard proof of this fact, at least if H has an element of infinite order. More precisely, it is well-known that a countable abelian group either has an element of infinite order or is a direct sum $\bigoplus_p A_p$ where the A_p are abelian p-groups, p prime. If H has an element of infinite order, or if all the A_p have finite exponent, then the methods of [35] should generalise. However, the case where at least one A_p has elements of unbounded order seems more complicated. Once this step done, we will look at virtually abelian groups G. In this case, there exists a finite subset $T \subset G$ and an abelian subgroup H such that $G = \langle H, T\rangle$. By combining carefully the results on countable abelian groups and on finitely generated groups, one might hope for a proof of Conjecture 15 for virtually countable groups.

We now look at the non-virtually abelian case. The idea is to generalise results of [33] which uses random walks and some ideas of Tointon [47] about commuting probabilities. Results of [47] are written for random walk (X_n) of law (μ^{*n}) where μ is a symmetric measure of finite generating support containing $\{1\}$. For the countable case, we plan to generalise Tointon's result to the following context. Let $S = \{s_1, s_1^{-1}, s_2, s_2^{-1}, \dots\}$ and let μ_n be the uniform measure on $\{1, s_1^{\pm 1}, s_2^{\pm 1}, \dots, s_n^{\pm 1}\}$. We then look at the sequence of random variables (X_n) of law (μ_n^{*n}). If S is finite, then for $n \geq |S|$ this is the classic random walk. If S is infinite, this is not anymore a random walk, but this looks like sufficiently like it so it is reasonable to hope to generalise results of [47] and [33] to this context.

2.4 Relevance and impact

On one hand, the original question of which groups can be represented as an automorphism groups of a suitably regular graph is itself of great interest and has attired the attention of many mathematicians. It took 10 years to solve this question for finite groups, and then a waiting of 40 years before having the solution for finitely generated groups. I strongly believe that we have now the right tools to attack it and to hope for a full resolution of it in the next years. Observe that the (easier) analog for directed graphs is known to be true for any groups since 1980.

On the other hand, the technics used to attack this problem turned out to be powerful and versatile. Indeed, not only the articles [34, 33] solved the original problem for finitely generated groups, the same proofs gave essentially for free the results for the directed variant (the generating set is S not necessary symmetric and $\vec{\mathrm{Cay}}(G;S)$ is a digraph) and the oriented variant (for any $x \in S$, the element x^{-1} does not belong to S) for finitely generated groups. The directed variant was already solved by Babai [1], but with the caveat of always providing an infinite generating set, while [34, 33] gave a finite generating set. For the oriented variant, this solved a conjecture raised by Babai in [34, 33]. Observe that the proof of the oriented variant for finite groups was only obtained in 2018 and use the classification of finite simple groups [37]. While in the directed case there is nothing more to prove for groups that are not finitely generated, the oriented version is still open. Our plan to prove Watkin's conjecture for countable groups will probably give in the same time a proof for the oriented version of the conjecture. Other consequences of the results of [34, 33] include rigidity phenomenon for coverings $\mathrm{Cay}(G;S) \to X$ where is either transitive or a Schreier graph of G, see [34] for more details. Once again, a solution to Conjecture 15 for countable groups is likely to come with generalisations of the aforementioned rigidity results.

Finally, the technics of [47] and [33] that we plan to generalise further to countable groups are themselves of interest. Indeed, they allow to push to infinite groups some results known to be true for finite groups, as "if half of the elements of a finite group G are of order 2, then G is abelian" that for finitely generated groups became "if $\mathbf{P}(g^2 = 1) > \frac{\sqrt{5}-1}{2}$, then G is virtually abelian (for a suitable probability measure $\mathbf{P}$)". It will be nice to obtain similar results for countable groups or even for arbitrary groups.

3 Limits of Rauzy digraphs

3.1 Current state of research in the field

De Bruijn graphs and their generalisations (spider-web graphs and Rauzy graphs) are ubiquitous objects used in many areas in science. On one hand, and from a pure mathematical point of view, they encode in finite objects the comportment of subshifts and are hence of great interest both for combinatorics and symbolic dynamics. They also enjoy really good connectivity properties and have remarkable percolation and spectral properties and, as we will see, are related to the Lamplighter group. On the other hand, their properties make them useful for applications. For example, spider-web networks were introduced by Ikeno in 1959 [25] in order to study systems of telephone exchanges. They were later shown to enjoy interesting properties in percolation, see [42], [43] and [44]. They are also an important tool for statistical physic [4]. As for Rauzy graphs, they are extensively used in bioinformatics to encode genome sequences [29].

At this point, let us recall the definition of de Bruijn graphs and their generalisations. For $k \geq 2$, and $N \geq 0$ we define the *de Bruijn digraph* $\vec{\mathcal{B}}_{k,N}$ as the graph with vertex set $\{x_1 \dots x_N \mid x_i \in \{1, \dots, k\}\} = \{1, \dots, k\}^N$ and for every vertex $x_1 \dots x_N$ and every $y \in \{1, \dots, k\}$ an arc from $x_1 \dots x_N$ to $x_2 \dots x_N y$. The undirected graph $\mathcal{B}_{k,N}$ is simply the underlying graph of $\vec{\mathcal{B}}_{k,N}$. If $M \geq 1$ is an integer, we define the *spider-web digraph* $\vec{\mathcal{S}}_{k,N,M}$ as the digraph with vertex set $\{1, \dots, k\}^N \times M$ and for every vertex $(x_1 \dots x_N, j)$ and every $y \in \{1, \dots, k\}$ an arc from $(x_1 \dots x_N, j)$ to $(x_2 \dots x_N y, j+1)$ where $j+1$ is taken modulo M. If Σ is a subshift over the alphabet $\{1, \dots, k\}$ the the *Rauzy digraph* $\vec{\mathcal{R}}_{k,N}$ is the subgraph of $\vec{\mathcal{B}}_{k,N}$ generated by all sequences $x_1 \dots x_N$ that appears as a subsequence of some biinfinite word in Σ. Finally, one can naturally define a spider-web version of Rauzy digraphs.

In 2012 [4], the phycisists Balram and Dhar studied the asymptotic properties of the sequence of spider-web graphs $\{\mathcal{S}_{k,N,M}\}$, for $k = 2$, where the $\{\mathcal{S}_{k,N,1}\}$ are the original de Bruijn graphs $\mathcal{B}_{k,N}$. In particular, they found, using an interesting approach based on symmetries, the spectra of graphs $\mathcal{S}_{2,N,M}$ and observed that they converge to a discrete limiting distribution as $M, N \to \infty$. This discrete distribution of the spectra is a rare phenomenon, which for Cayley graphs was only known to hold for Cayley graphs of the Lamplighter groups $\mathcal{L}_k = \mathbf{Z}/k\mathbf{Z} \wr \mathbf{Z} = (\bigoplus_{\mathbf{Z}} \mathbf{Z}/k\mathbf{Z}) \rtimes \mathbf{Z}$. It was hence natural to ask if this was only a coincidence, or if it followed from some deeper relationship between spider-web graphs and $\mathcal{L}_k$.

With my coauthors we solved this questions in 2016 [21], showing that the $\{\mathcal{S}_{k,N,M}\}$ converge (in the sense of Benjamini-Schramm) to the Cayley graphs of $\mathcal{L}_k$. Since the convergence of the graphs imply the convergence of the spectral measure, this fully explained the connection discovered by Balram and Dhar. Moreover, having a well-identified limiting object allow to better understand the asymptotic comportement of the family $\{\mathcal{S}_{k,N,M}\}$. But this relation is two ways and it also allows us to do computations on the finite graphs in $\{\mathcal{S}_{k,N,M}\}$ in order to obtain result for the infinite Cayley graph of $\mathcal{L}_k$.

While de Bruijn graphs are an important object in combinatorics, they are sometimes too restrictive and we often (especially for applications) look at their generalisation: Rauzy graphs. Basically, de Bruijn graphs encode finite sequences over some given alphabet (for example $\{A, C, G, T\}$), where every subsequence is allowed. For Rauzy graphs, we are allowed to specify a set of forbidden subsequences, that is to look at a subshift. It is therefore of interest (both from a pure mathematical point of view and for applications) to generalise to Rauzy graphs the results known to hold for de Bruijn graphs.

The study of the limit of Rauzy graphs only started recently. It was initiated in my thesis [31] where some partial results for subshifts of finite type were obtained by combinatorics methods. Some of the results of [21] and [31] were later refined in [30]. Finally, in his master thesis [9] take care of the Rauzy graphs of subshifts of complexity at most linear and showed that in this case the $R_{\Sigma,N}$ converge to a biinfinite line.

3.2 Current state of personal research

With my co-authors we developed in 2016 a method that leads to the full understanding of the infinite discrete model for the $\{\mathcal{S}_{k,N,M}\}$, including its spectral characteristics, via finite approximations, using the notion of Benjamini-Schramm limit of graphs that has lately become very important in probability theory. A remarkable feature of the model that we discovered is that it is related to one of the most interesting and important test-cases in combinatorial group theory, both algebraically and from the spectral and probabilistic viewpoints, the lamplighter groups $\mathcal{L}_k$.

Theorem 22 ([21])**.** *The spider-web graphs converge to the Cayley graphs of the Lamplighter group $\mathcal{L}_k$ (for some natural generating set S_k of $\mathcal{L}_k$). More precisely, we have that $\lim_{N,M\to\infty} \mathcal{S}_{k,N,M} = \mathrm{Cay}(\mathcal{L}_k, S_k)$, but also that $\lim_{N\to\infty} \mathcal{S}_{k,N,M} = \mathrm{Cay}(\mathcal{L}_k, S_k)$ for any fixed M.*

In particular, the spectral measure of the $B_{k,n}$ converge to the spectral measure on the Cayley graph of $\mathcal{L}_k$. The same is true for any "reasonable quantity" associated to the $B_{k,n}$.

In order to prove the above theorem, we proved the following two important structural results:

Theorem 23 ([21])**.** *The directed spider-web graph $\vec{\mathcal{S}}_{k,N,M}$ is isomorphic to the tensor product $\vec{\mathcal{B}}_{k,N} \otimes \vec{C}_M$ of a directed de Bruijn graph with a directed cycle on M vertices.*

Theorem 24 ([21])**.** *The de Bruijn graphs $\mathcal{B}_{k,N}$ are isomorphic to Schreier graphs of the canonical actions of $\mathcal{L}_k$ on the k-regular rooted tree.*

The description as a tensor product allowed us to concentrate our attention on the original de Bruijn graphs, while the description these latter as Schreier graphs imply the convergence to the Cayley graphs of $\mathcal{L}_k$.

For a general subshift and the corresponding Rauzy graphs, the analogous of Theorem 23 remains true. However, Theorem 24 cannot hold as the Rauzy graphs are not anymore regular. We hence need a new candidate for the limit, as well as new technics to show the convergence. Let us recall that in general the Benjamini-Schramm limit of a family of finite graphs is not necessarily a graph, but more generally a probability measure on the space of rooted graphs.

At this point I used the following key observation: the Cayley graph of $\mathcal{L}_k$ is isomorphic to an horocyclic product of two $(k+1)$-regular tree. Using combinatorics methods and the Perron-Froebenius theorem I was able to prove the following

Theorem 25 ([31])**.** *Let Σ be a subshift of finite type (subject to some technical conditions) and let $R_{\Sigma,N}$ be the associated Rauzy graphs. Then the limit of $R_{\Sigma,N}$ exist and is supported on horocyclic products of (non necessarily regular) trees.*

More precisely, there exists an explicit map g from Σ to the space of rooted graphs such that for every $\omega \in \Sigma$ the rooted graph $g(\omega)$ is an horocyclic product of trees and such that $\lim_N R_{\Sigma,N}$ is supported on $g(\Sigma)$.

3.3 Detailed research plan

The main objective of this project is to identify the limit of the family of Rauzy digraphs associated to a given subshift. Put in other words, we aim to extend the result of [21] to general Rauzy digraphs.

Question 26. *Let $\Sigma \subseteq \{1, \dots, k\}^{\mathbf{Z}}$ be a subshift. What is the limit of the Rauzy graphs $R_{\Sigma,n}$?*

We already know that the limit, if it exists, will be a probability measure on the space of rooted graphs. In a first step we will show that this probability measure is supported on horocyclic product of trees, that is we plan to extend Theorem 25 to any subshift. The proof of this phenomenon will be by dychotomy. For subshifts of finite type, this is mostly combinatoric and we will use the Perron-Frobenius Theorem. The idea is to follow the general ideas from [31], but to pay more attention to the technical details to be able to remove the "technical conditions" from the Theorem's statement. For subshifts of subexponential complexity, Perron-Frobenius Theorem is not anymore available, but combining dynamical and geometrical arguments we should be able to show that the graphs are sparse which in our case will be enough to identify the limit. This second part is a common project with T. Nagnibeda, A. Skripchenko and G. Veprev. This dichotomy leaves open the case of subshifts of exponential complexity that are not of finite type. New ideas will be required to tackle this last case, and the entropy might play an important role.

On a second step, we plan to identify $\lim_N \mathcal{R}_{\Sigma,N}$ as the pushforward of some measure associated to the subshift. For subshifts of finite type, it would be the unique measure of maximal entropy. For subshifts of sub-exponential complexity the situation is less clear and the corresponding measure has yet to be determined. Preliminaries computations show that $\lim_N \mathcal{R}_{\Sigma,N}$ should be the pushfoward by the map g described in [31], so we already have a good candidate both for the map and for the measure (for subshifts of finite type). This part is a common project with V. Kaimanovich and T. Nagnibeda.

Finally, we will use the above results to do concrete computations, as for example with the spectral eigenvalues and the spectral measures. One specific application that interest me is the computation of the *complexity* of the finite graphs under consideration, where the complexity $\tau(\Gamma)$ of a finite graph is the number of coverings trees. This is a natural invariant which is often studied in the cas of finite graphs. One particular interest of this invariant is that it admits an algebraic characterisation: $\tau(\Gamma)\cdot|V(\Gamma)|$ is equal to the product of the non-zero eigenvalues of the Laplacian of Γ, where $|V(\Gamma)|$ is the number of vertices in Γ. For infinite family of finite graphs one are more specifically interested in the asymptotic complexity $\tau_{\text{as}} \coloneqq \lim_n \frac{\log\tau(\Gamma_n)}{|V(\Gamma_n)|}$. R. Lyons showed in 2003 that if the Γ_n converge to some limit Γ, then τ_{as} exists and can be computed directly from Γ using random walks. Moreover, under some mild technical hypothesis (that the de Bruijn graphs satisfy), $\tau_{\text{as}} = -\zeta_\Gamma'(0)$ where ζ_Γ is the spectral zeta function of Γ. Direct computations show that for de Bruinj graphs we have

$$\tau_{\text{as}} = \log(k) - (k-1)^2 \frac{\mathrm{d}}{\mathrm{d}s} \operatorname{Li}_s\big(\frac{1}{k}\big)\big|_{s=0}$$

where $\operatorname{Li}_s(z)$ is the polylogarithm. It is well-known that for $\operatorname{Re}(s) > 1$ the function $\operatorname{Li}_s(1)$ is the classical Riemann zeta function. We hence plan to investigate in more detials the relation between the polylogarithm (and the Riemann zeta function) and the spectral zeta function on the de Bruijn graphs.

Further developments include the study of Rauzy graphs for subshifts over $\mathbf{Z}^d$ or over a general group, as well as the study of subshifts over a countable alphabet.

3.4 Relevance and impacts

Answering Question 26 will be really interesting from a pure mathematics point of view, but also for the applications. On one hand, this will provide a nice approximation by finite graphs of subshifts. This will hence provide us with a better understanding of symbolic dynamic. It will also provide us with a link between subshifts and horocyclic products of trees (also known as Diestel-Leader graphs), two subjects that underwent great development these recent years.

On the other hand, computing the limit of the $\mathcal{R}_{k,N}$ will allow us to have a full understanding of the asymptotic comportment of the associated quantities (as the spectral measure or the spectral eigenvalues for example), which is specifically important for applications.

References

[1] László Babai. Infinite digraphs with given regular automorphism groups. *J. Combin. Theory Ser. B*, 25(1):26–46, 1978.

[2] László Babai. Finite digraphs with given regular automorphism groups. *Period. Math. Hungar.*, 11(4):257–270, 1980.

[3] László Babai and Christopher D. Godsil. On the automorphism groups of almost all Cayley graphs. *European J. Combin.*, 3(1):9–15, 1982.

[4] Ajit C. Balram and Deepak Dhar. Non-perturbative corrections to mean-field critical behavior: the spherical model on a spider-web graph. *J. Phys. A*, 45(12):125006, 14, 2012.

[5] Laurent Bartholdi and Rostislav I. Grigorchuk. On parabolic subgroups and Hecke algebras of some fractal groups. *Serdica Math. J.*, 28(1):47–90, 2002.

[6] Laurent Bartholdi, Olivier Siegenthaler, and Pavel Zalesskii. The congruence subgroup problem for branch groups. *Israel J. Math.*, 187:419–450, 2012.

[7] Khalid Bou-Rabee, Paul-Henry Leemann, and Tatiana Nagnibeda. Weakly maximal subgroups in regular branch groups. *J. Algebra*, 455:347–357, 2016.

[8] Edward Dobson, Pablo Spiga, and Gabriel Verret. Cayley graphs on abelian groups. *Combinatorica*, 36(4):371–393, 2016.

[9] Blair Drummond. Limits of rauzy graphs of low-complexity words. Master's thesis, University of Ottawa, 2019.

[10] Paul Erdős and Alfréd Rényi, editors. *Combinatorial theory and its applications. I-III.* North-Holland Publishing Co., Amsterdam-London, 1970. Colloquia Mathematica Societatis János Bolyai, 4.

[11] Gustavo A. Fernández-Alcober and Amaia Zugadi-Reizabal. GGS-groups: order of congruence quotients and Hausdorff dimension. *Trans. Amer. Math. Soc.*, 366(4):1993–2017, 2014.

[12] Dominik Francoeur and Paul-Henry Leemann. Subgroup induction property for branch groups. *arXiv e-prints*, page arXiv:2011.13310, November 2020.

[13] Giovanni Gandini. Cohomological invariants and the classifying space for proper actions. *Groups Geom. Dyn.*, 6(4):659–675, 2012.

[14] Alejandra Garrido. Abstract commensurability and the Gupta-Sidki group. *Groups Geom. Dyn.*, 10(2):523–543, 2016.

[15] Alejandra Garrido. On the congruence subgroup problem for branch groups. *Israel J. Math.*, 216(1):1–13, 2016.

[16] Christopher D. Godsil. GRRs for nonsolvable groups. In *Algebraic methods in graph theory, Vol. I, II (Szeged, 1978)*, volume 25 of *Colloq. Math. Soc. János Bolyai*, pages 221–239. North-Holland, Amsterdam-New York, 1981.

[17] R. I. Grigorchuk and S. N. Sidki. The group of automorphisms of a 3-generated 2-group of intermediate growth. *Internat. J. Algebra Comput.*, 14(5-6):667–676, 2004. International Conference on Semigroups and Groups in honor of the 65th birthday of Prof. John Rhodes.

[18] Rostislav I. Grigorchuk. On Burnside's problem on periodic groups. *Funktsional. Anal. i Prilozhen.*, 14(1):53–54, 1980.

[19] Rostislav I. Grigorchuk. Degrees of growth of finitely generated groups and the theory of invariant means. *Izv. Akad. Nauk SSSR Ser. Mat.*, 48(5):939–985, 1984.

[20] Rostislav I. Grigorchuk, P.-H. Leemann, and T. V. Nagnibeda. Finitely generated subgroups of branch groups and subdirect products of just infinite groups. *Izv. Ross. Akad. Nauk Ser. Mat.*, 85(6):104–125, 2021.

[21] Rostislav I. Grigorchuk, Paul-Henry Leemann, and Tatiana Nagnibeda. Lamplighter groups, de Brujin graphs, spider-web graphs and their spectra. *J. Phys. A*, 49(20):205004, 35, 2016.

[22] Rostislav I. Grigorchuk and John S. Wilson. A structural property concerning abstract commensurability of subgroups. *J. London Math. Soc. (2)*, 68(3):671–682, 2003.

[23] Rostistlav I. Grigorchuk and Tatiana Nagnibeda. On subgroup structure of a 3-generated 2-group of intermediate growth. In *Profinite and Asymptotic Group Theory (MFO, Oberwolfach, Germany, June 2008)*, volume 28/2008, pages 1568–1571, Oberwolfach, Germany, 2008. Mathematisches Forschungsinstitut Oberwolfach.

[24] D. Hetzel. *Über reguläre graphische Darstellung von auflösbaren Gruppen.* PhD thesis, Technische Universität Berlin, 1976.

[25] Nobuichi Ikeno. A limit on crosspoint numbers. *IRE Trans. Inform. Theory*, 5:18–196, 1959.

[26] Wilfried Imrich. Graphen mit transitiver Automorphismengruppe. *Monatsh. Math.*, 73:341–347, 1969.

[27] Wilfried Imrich. On graphs with regular groups. *J. Combinatorial Theory Ser. B*, 19(2):174–180, 1975.

[28] Wilfried Imrich and Mark E. Watkins. On automorphism groups of Cayley graphs. *Period. Math. Hungar.*, 7(3-4):243–258, 1976.

[29] Z. Iqbal, M. Caccamo, I. Turner, P. Flicek, and G. McVean. De novo assembly and genotyping of variants using colored de bruijn graphs. *Nature Genetics*, 44, 2012.

[30] Vadim A. Kaimanovich. Circular slider graphs: de bruijn, kautz, rauzy, lamplighters and spiders. *arXiv: Dynamical Systems*, 2018.

[31] Paul-Henry Leemann. *On Subgroups and Schreier Graphs of Finitely Generated Groups.* PhD thesis, Université de Genève, 2016.

[32] Paul-Henry Leemann. Weakly maximal subgroups of branch groups. *arXiv e-prints*, page arXiv:1910.06399, Oct 2020.

[33] Paul-Henry Leemann and Mikael de la Salle. Cayley graphs with few automorphisms: the case of infinite groups. *Accepted in Annales Henri Lebesgue(2021). arXiv e-prints*, page arXiv:2010.06020, October 2020.

[34] Paul-Henry Leemann and Mikael de la Salle. Cayley graphs with few automorphisms. *J. Algebraic Combin.*, 53(4):1117–1146, 2021.

[35] Paul-Henry Leemann and Mikael de la Salle. Most rigid representation and Cayley index of finitely generated groups. *arXiv e-prints*, page arXiv:2105.02326, May 2021.

[36] Anatoly I. Mal'cev. On homomorphisms onto finite groups. *Ivanov. Gos. Ped. Inst. Ucen. Zap.*, 18:49–60, 1958. English translation: Transl., Ser. 2, Am. Math. Soc. 119, 67-79, 1983.

[37] Joy Morris and Pablo Spiga. Classification of finite groups that admit an oriented regular representation. *Bull. Lond. Math. Soc.*, 50(5):811–831, 2018.

[38] Joy Morris, Pablo Spiga, and Gabriel Verret. Automorphisms of Cayley graphs on generalised dicyclic groups. *European J. Combin.*, 43:68–81, 2015.

[39] Volodymyr Nekrashevych. *Self-similar groups*, volume 117 of *Mathematical Surveys and Monographs.* American Mathematical Society, Providence, RI, 2005.

[40] Lewis A. Nowitz and Mark E. Watkins. Graphical regular representations of non-abelian groups. I, II. *Canad. J. Math.*, 24:993–1008; ibid. 24 (1972), 1009–1018, 1972.

[41] Ekaterina L. Pervova. Everywhere dense subgroups of a group of tree automorphisms. *Tr. Mat. Inst. Steklova*, 231(Din. Sist., Avtom. i Beskon. Gruppy):356–367, 2000.

[42] Nicholas Pippenger. The blocking probability of spider-web networks. *Random Structures Algorithms*, 2(2):121–149, 1991.

[43] Nicholas Pippenger. The asymptotic optimality of spider-web networks. *Discrete Appl. Math.*, 37/38:437–450, 1992.

[44] Nicholas Pippenger. The linking probability of deep spider-web networks. *SIAM J. Discrete Math.*, 20(1):143–159 (electronic), 2006.

[45] Matatyahu Rubin. Locally moving groups and reconstruction problems. In *Ordered groups and infinite permutation groups*, volume 354 of *Math. Appl.*, pages 121–157. Kluwer Acad. Publ., Dordrecht, 1996.

[46] Rachel Skipper and Phillip Wesolek. On the Cantor-Bendixson rank of the Grigorchuk group and the Gupta-Sidki 3 group. *J. Algebra*, 555:386–405, 2020.

[47] Matthew C. H. Tointon. Commuting probabilities of infinite groups. *J. Lond. Math. Soc. (2)*, 101(3):1280–1297, 2020.

[48] Mark E. Watkins. On the action of non-Abelian groups on graphs. *J. Combinatorial Theory Ser. B*, 11:95–104, 1971.

[49] Mark E. Watkins. On graphical regular representations of $C_n \times Q$. In *Graph theory and applications (Proc. Conf., Western Michigan Univ., Kalamazoo, Mich., 1972; dedicated to the memory of J. W. T. Youngs)*, pages 305–311. Lecture Notes in Math., Vol. 303. Springer, Berlin, 1972.

[50] Mark E. Watkins. Graphical regular representations of alternating, symmetric, and miscellaneous small groups. *Aequationes Math.*, 11:40–50, 1974.

[51] Mark E. Watkins. Graphical regular representations of free products of groups. *J. Combinatorial Theory Ser. B*, 21(1):47–56, 1976.

[52] Henry Wilton. Hall's theorem for limit groups. *Geom. Funct. Anal.*, 18(1):271–303, 2008.

[53] Binzhou Xia and Shasha Zheng. Asymptotic enumeration of graphical regular representations. *arXiv e-prints*, page arXiv:2212.01875, December 2022.