

**Exercise 1.**

Show that axiom (Mo) in the definition of a metric space is not needed, i.e. assuming that axioms (M1) to (M4) hold, prove that (Mo) has to hold.

*Solution.* Putting  $z = x$  into (M4) we obtain that  $d(x, x) \leq d(x, y) + d(y, x)$ , for all  $x, y \in X$ . Using (M2) and (M3) implies that  $0 \leq 2d(x, y) \implies d(x, y) \geq 0$ , for all  $x, y \in X$ . ■

**Exercise 2.**

Prove the triangle inequality (M4) for Euclidean  $n$ -space:

$$d_2(x, z) \leq d_2(x, y) + d_2(y, z), \quad \forall x, y, z \in \mathbf{R}^n,$$

by first letting  $r_i := x_i - y_i$  and  $s_i := y_i - z_i$ . So we need to prove

$$\left( \sum_{i=1}^n (r_i + s_i)^2 \right)^{\frac{1}{2}} \leq \left( \sum_{i=1}^n r_i^2 \right)^{\frac{1}{2}} + \left( \sum_{i=1}^n s_i^2 \right)^{\frac{1}{2}}.$$

First show that this is equivalent to **Cauchy's Inequality**:

$$\left( \sum_{i=1}^n r_i s_i \right)^2 \leq \sum_{i=1}^n r_i^2 \sum_{i=1}^n s_i^2.$$

Now prove Cauchy's Inequality. *Hint: You may want to consider that, for all  $\lambda \in \mathbf{R}$ ,  $\sum_{i=1}^n (r_i + \lambda s_i)^2 \geq 0$ .*

*Solution.* To obtain Cauchy's Inequality, square both sides of the equation to get

$$\sum_{i=1}^n (r_i + s_i)^2 = \sum_{i=1}^n r_i^2 + 2 \sum_{i=1}^n r_i s_i + \sum_{i=1}^n s_i^2 \leq \sum_{i=1}^n r_i^2 + 2 \left( \sum_{i=1}^n r_i^2 \right)^{\frac{1}{2}} \left( \sum_{i=1}^n s_i^2 \right)^{\frac{1}{2}} + \sum_{i=1}^n s_i^2,$$

which yields the result. To prove Cauchy's Inequality notice that, by the hint,

$$f(\lambda) := \lambda^2 \sum_{i=1}^n s_i^2 + 2\lambda \sum_{i=1}^n r_i s_i + \sum_{i=1}^n r_i^2 \geq 0.$$

Which implies that the quadratic equation  $f(\lambda) = 0$  cannot have two distinct real roots. So the "discriminant"  $b^2 - 4ac$  must be non-positive which gives the result.

For an alternative proof of Cauchy's Inequality, one can use the Cauchy-Schwarz inequality for the standard dot product on  $\mathbf{R}^n$ , see MTH107 Advanced Linear Algebra. ■

**Exercise 3.**

Show that the following function defines a metric on  $\mathbf{R}^n$ :

$$d_\infty(x, y) := \max\{|x_1 - y_1|, \dots, |x_n - y_n|\}.$$

*Solution.* The axioms (M0) obviously holds. For (M1) and (M2),  $d_\infty(x, y) = 0$  if and only if  $|x_i - y_i| = 0$  for all  $1 \leq i \leq n$ , if and only if  $x = y \in \mathbf{R}^n$ . (M3) follows from  $|x - y| = |y - x|$ . For (M4), for each  $1 \leq i \leq n$ ,

$$|x_i - z_i| \leq |x_i - y_i| + |y_i - z_i| \leq d_\infty(x, y) + d_\infty(y, z),$$

so  $d_\infty(x, z) \leq d_\infty(x, y) + d_\infty(y, z)$ . ■

#### Exercise 4.

Determine whether or not the following functions  $d$  define metrics on the indicated sets:

- $d(x, y) = |x^2 - y^2|$ , for  $x, y \in \mathbf{R}$ ;
- $d(x, y) = |x^2 - y^2|$ , for  $x, y \in [0, \infty) \subseteq \mathbf{R}$ ;
- $d(x, y) = (x_1 - y_1)^2 + (x_2 - y_2)^2$ , for  $x, y \in \mathbf{R}^2$ .

*Solution.* a. No, since  $d(x, -x) = 0$  for  $x \neq 0$  which goes against (M2).

b. Yes. For the triangle inequality  $|x^2 - z^2| \leq |x^2 - y^2| + |y^2 - z^2|$  for all  $x, y, z \geq 0$ .

c. No. Notice that

$$4 = d((2, 0), (0, 0)) > d((2, 0), (1, 0)) + d((1, 0), (0, 0)) = 1 + 1 = 2;$$

so the triangle inequality fails. ■

#### Exercise 5.

Show that the *British Railway*<sup>a</sup> metric on  $\mathbf{R}^2$ , given by

$$d(x, y) := \begin{cases} d_2(x, y), & \text{if } x \text{ and } y \text{ are linearly dependent;} \\ d_2(x, 0) + d_2(0, y), & \text{otherwise,} \end{cases}$$

satisfies the axioms of a metric space, where  $d_2$  is the Euclidean metric on  $\mathbf{R}^2$ .

<sup>a</sup>Also known as the French Railway metric.

*Solution.* The axioms (M0)–(M3) are easy to check. For (M4), first notice that  $d_2(x, y) \leq d_2(x, 0) + d_2(0, y)$ , for all  $x, y \in \mathbf{R}^2$ . If  $x$  and  $z$  are linearly dependent then

$$d(x, z) = d_2(x, z) \leq d_2(x, y) + d_2(y, z) \leq d(x, y) + d(y, z).$$

If  $x$  and  $z$  are linearly independent, then either  $\{x, y\}$  and  $\{y, z\}$  are both linearly independent sets or  $y$  is dependent on either  $x$  or  $z$ . In the first case, (M4) is equivalent to  $d_2(0, y) \geq 0$ , which always holds, and in the second case (if we assume that  $x$  and  $y$  are dependent) then

$$d(x, z) = d_2(x, 0) + d_2(0, z) \leq d_2(x, y) + d_2(y, 0) + d_2(0, z) = d(x, y) + d(y, z). \quad \blacksquare$$

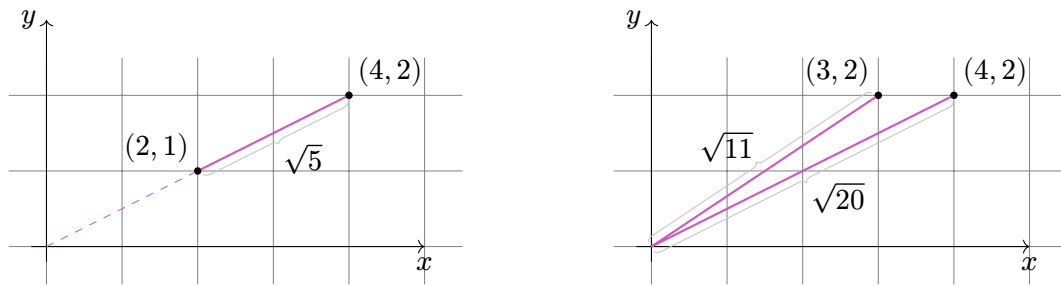


Figure 1: Shortest paths for the British railway distance between two points in the plane.

We have  $d((2, 1), (4, 2)) = \sqrt{5}$  while  $d((3, 1), (4, 2)) = \sqrt{11} + \sqrt{20}$ .

### Exercise 6.

Given three points  $x, y, z \in X$  in a metric space  $(X, d)$ , show that

$$|d(x, z) - d(y, z)| \leq d(x, y).$$

*Solution.* It's equivalent to  $-d(x, y) \leq d(x, z) - d(y, z) \leq d(x, y)$  and each inequality is equivalent to the triangle inequality. ■

### Exercise 7.

Show that the following functions between metric spaces are continuous.

- The identity function:  $\text{Id}_X: (X, d) \rightarrow (X, d)$ , where  $\text{Id}_X(x) := x$ , for all  $x \in X$ .
- Inclusion:  $i: (W, d|_W) \rightarrow (X, d)$ , where  $W \subseteq X$  and  $i(w) := w$ , for all  $w \in W$ .
- Constant function:  $f: (X, d_X) \rightarrow (Y, d_Y)$ , where  $f(x) := y_0$ , for all  $x \in X$  and some fixed  $y_0 \in Y$ .
- Show that every isometry is continuous.

*Solution.* a. Take  $\delta = \varepsilon$ .

b. Take  $\delta = \varepsilon$ .

c. Take any  $\delta > 0$ .

d. Since  $d_Y(f(x), f(a)) = d_X(x, a)$  we can take  $\delta = \varepsilon$ . ■

**Exercise 8.**

Suppose that  $f: (X, d_X) \rightarrow (Y, d_Y)$  and  $g: (Y, d_Y) \rightarrow (Z, d_Z)$  are continuous functions. Show that the composition

$$g \circ f: (X, d_X) \rightarrow (Z, d_Z),$$

where  $(g \circ f)(x) := g(f(x))$ , for all  $x \in X$ , is continuous.

*Solution.* Let  $\varepsilon > 0$  and  $a \in X$ . We need to find  $\delta > 0$  such that

$$d_X(x, a) < \delta \implies d_Z(g(f(x)), g(f(a))) < \varepsilon.$$

Since  $g$  is continuous at  $f(a) \in Y$  there exists  $\delta_g > 0$  such that

$$d_Y(y, f(a)) < \delta_g \implies d_Z(g(y), g(f(a))) < \varepsilon.$$

Also,  $f$  is continuous at  $a \in X$  so there exists  $\delta_f > 0$  such that

$$d_X(x, a) < \delta_f \implies d_Y(f(x), f(a)) < \delta_g.$$

We can then take  $\delta = \delta_f$  to get what we need by setting  $y = f(x)$ . ■

**Exercise 9.**

Let  $(X, d_X)$  be a metric space and fix  $x_0 \in X$ . Show that the function

$$f: (X, d_X) \implies (\mathbf{R}, d_{\mathbf{R}}),$$

given by  $f(x) := d_X(x, x_0)$ , for all  $x \in X$ , is continuous (where  $d_{\mathbf{R}}$  is the usual metric on  $\mathbf{R}$ ).

*Solution.* Let  $\varepsilon > 0$  and  $a \in X$ . Then we need to find  $\delta > 0$  such that  $d_X(x, a) < \delta \implies |f(x) - f(a)| < \varepsilon$ . This is equivalent to

$$d_X(a, x_0) - \varepsilon < d_X(x, x_0) < d_X(a, x_0) + \varepsilon.$$

Since for any  $x \in X$ ,  $d_X(x, x_0) \leq d_X(x, a) + d_X(a, x_0)$  and  $d_X(a, x_0) \leq d_X(a, x) + d_X(x, x_0)$  we can just take  $\delta = \varepsilon$ . ■

Exercises marked with a \* are slightly harder bonus questions.

### Exercise 1.

- Compute  $d_{\text{sup}}(x^2, x)$  in  $\mathcal{C}[0, 1]$  and  $d_1(x^2, x)$  in  $\mathcal{L}_1[0, 1]$ .
- Assume that  $1 < a < b$  in  $\mathbf{R}$ . Compute  $d_{\text{sup}}(x^3, 2 - x^2)$  in  $\mathcal{C}[a, b]$  and  $d_1(x^3, 2 - x^2)$  in  $\mathcal{L}_1[a, b]$ .

*Solution.* a. On  $[0, 1]$  we have  $|x^2 - x| = x - x^2$ . Moreover,  $d(x - x^2)/dx = 1 - 2x$  is 0 if and only if  $x = 1/2$ . So  $x = 1/2$  is a local extremum. Moreover  $d^2(x - x^2)/dx^2 = -2 < 0$  when  $x = 1/2$ , so  $x = 1/2$  is a local maximum. We conclude  $d_{\text{sup}}(x^2, x) = \sup_{x \in [0, 1]}(x - x^2) = 1/2 - 1/4 = 1/4$  in  $\mathcal{C}[0, 1]$ .

For  $d_1$  a direct computation gives us  $d_1(x^2, x) = \int_0^1 (x - x^2) dx = [x^2/2 - x^3/3]_0^1 = 1/2 - 1/3 = 1/6$  in  $\mathcal{L}_1[0, 1]$ .

- Since  $1 < a < b$ , we have  $x^3 + x^2 > 2$  on  $[a, b]$  and it attains its absolute maximum at  $x = b$ . So  $d_{\text{sup}}(x^3, 2 - x^2) = b^3 + b^2 - 2$  in  $\mathcal{C}[a, b]$ . In  $\mathcal{L}_1[a, b]$

$$d_1(x^3, 2 - x^2) = \int_a^b |x^3 + x^2 - 2| dx = \left[ \frac{x^4}{4} + \frac{x^3}{3 - 2x} \right]_a^b,$$

which reduces to  $\frac{b^4 - a^4}{4} + \frac{b^3 - a^3}{3} - 2(b - a)$ . ■

### Exercise 2.

Explain why neither of the following functions define a metric on the set of continuous functions  $[0, 1] \rightarrow \mathbf{R}$ :

- $d(f, g) = |f(1/2) - g(1/2)|$ ;
- $d(f, g) = \sup_{x \in [0, 1]} |f'(x) - g'(x)|$ , where  $f'(x) := df(x)/dx$ .

*Solution.* a. Let  $f(x) = x$  and  $g(x) = 1 - x$ . Then  $d(f, g) = 0$  so axiom (M2) fails.

- If  $f$  or  $g$  fails to be differentiable for some  $x \in [0, 1]$ , then  $d(f, g)$  is not defined! But even so, the examples  $f(x) = 1$  and  $g(x) = 0$  show that  $d(f, g) = 0$  so (M2) fails again. ■

### Exercise 3.

Let  $X := \{0, 1\}^\infty$  be the set of all infinite **binary sequences**  $x = x_0x_1x_2 \dots$ , where  $x_n \in \{0, 1\}$ , for  $n \geq 0$ ; so  $x = 100110111 \dots$  is a typical element. Show that the following are metrics on  $X$ .

a.

$$d_{\min}(x, y) := \begin{cases} 0, & \text{if } x = y; \\ \frac{1}{2^n}, & \text{if } n = \min\{m \mid x_m \neq y_m\}. \end{cases}$$

b.

$$d^*(x, y) := \sum_{j=0}^{\infty} \frac{|x_j - y_j|}{2^j}.$$

Find two elements  $x, y \in X$  such that  $d_{\min}(x, y) \neq d^*(x, y)$ .

*Solution.* a. (M1)–(M3) are satisfied automatically. For (M4), let  $d_{\min}(x, y) = 1/2^p$  and  $d_{\min}(y, z) = 1/2^q$ . So  $d_{\min}(x, z) = 1/2^r$ , where  $r = \min\{p, q\}$  if  $p \neq q$  and  $r > \min\{p, q\}$  if  $p = q$ . In both cases  $1/2^r \leq 1/2^p + 1/2^q$ , as required. If  $x = y$  or  $y = z$ , the proof is trivial.

b. Notice that the right hand side is bounded above by 2 and therefore converges.

The axioms (M1)–(M4) then follow by elementary properties of convergent series. Taking  $x = 0100 \dots$  and  $y = 100 \dots$  gives  $d_{\min}(x, y) = 1$  and  $d^*(x, y) = 3/2$ . ■

#### \*Exercise 4.

Let  $X$  be the set of all closed intervals  $[a, b]$  of  $\mathbf{R}$ . Show that

$$d([a, b], [c, d]) := \max\{|c - a|, |d - b|\}$$

is a metric on  $X$ .

Now consider the metric space  $(H, d_{\infty})$ , where

$$H := \{(x_1, x_2) \in \mathbf{R}^2 \mid x_1 \leq x_2\},$$

and show that it is isometric to  $(X, d)$ .

*Solution.* The axioms (M1) and (M2) are satisfied because  $|c - a| = |d - b| = 0$  if and only if  $c = a$  and  $d = b$ . Axiom (M3) is satisfied because  $|x - y| = |y - x|$ , for all  $x, y \in \mathbf{R}$ . For (M4), consider intervals  $I = [a, b]$ ,  $J = [c, d]$  and  $K = [e, f]$  in  $X$ . Then

$$d(I, J) = \max\{|c - a|, |d - b|\} \leq \max\{|c - e| + |e - a|, |d - f| + |f - b|\},$$

since  $|x - z| \leq |x - y| + |y - z|$ , for all  $x, y, z \in \mathbf{R}$ . Also,  $|c - e| + |e - a|$  and  $|d - f| + |f - b|$  are both less than or equal to  $\max\{|e - a|, |f - b|\} + \max\{|e - c|, |f - d|\}$  so

$$d(I, J) \leq \max\{|e - a|, |f - b|\} + \max\{|e - c|, |f - d|\} = d(I, K) + d(K, J).$$

We define a function  $f: X \rightarrow H$  by  $f([a, b]) = (a, b)$ . It is easy to see that  $f$  is a bijection. Also, since

$$d_{\infty}((x_1, x_2), (y_1, y_2)) = \max\{|y_1 - x_1|, |y_2 - x_2|\},$$

$f$  gives an isometry between  $(X, d)$  and  $(H, d_\infty)$ . ■

### Exercise 5.

Let  $\mathcal{X}$  be the space of bounded sequences of real numbers. Show that the map  $\|\cdot\|_\infty : \mathcal{X} \rightarrow \mathbf{R}$  defined by  $\|a\|_\infty := \sup_{n \in \mathbf{N}} |a_n|$  is a norm on  $\mathcal{X}$ .

*Solution.* Before verifying the norm axioms, we need to check that the map  $\|\cdot\|_\infty$  is well-defined. This is the case as  $a$  bounded implies that the supremum exists (and belongs to  $\mathbf{R}$ ).

“(No)” It directly follows from the definition that  $\|a\|_\infty \geq 0$  for all  $a$ .

“(N1) and (N2)”  $\|a\|_\infty = 0$  if and only if  $|a_n| = 0$  for all  $n$ , if and only if  $a$  is the zero sequence.

“(N3)”  $\|\lambda a\|_\infty = \sup_{n \in \mathbf{N}} |\lambda a_n| = \sup_{n \in \mathbf{N}} |\lambda| |a_n| = |\lambda| \sup_{n \in \mathbf{N}} |a_n| = |\lambda| \|a\|_\infty$ .

“(N4)” Let  $a$  and  $b$  be two sequences of real numbers. For every  $n$  we have  $|a_n + b_n| \leq |a_n| + |b_n|$ . This implies that for bounded sequences  $\|a + b\|_\infty = \sup_{n \in \mathbf{N}} |a_n + b_n| \leq \sup_{n \in \mathbf{N}} |a_n| + \sup_{n \in \mathbf{N}} |b_n| = \|a\|_\infty + \|b\|_\infty$ . ■

### Exercise 6.

Let  $\mathcal{Y}$  be the space of absolutely convergent sequences of real numbers. Show that the map  $\|\cdot\|_1 : \mathcal{Y} \rightarrow \mathbf{R}$  defined by  $\|a\|_1 := \sum_{n \in \mathbf{N}} |a_n|$  is a norm on  $\mathcal{Y}$ . (You may need to revise your properties of absolutely convergent series.)

*Solution.* By definition,  $\mathcal{Y}$  is the space of sequences  $a$  such that  $\|a\|_1$  is well-defined.

“(No)–(N2)” These are straightforward verifications, similar to Question 5

Before going further, let us recall the Rearrangement Theorem: The terms of an absolutely convergent series can be rearranged in any order, and the series will still converge to the same sum.

“(N3)”  $\|\lambda a\|_1 = \sum_{n \in \mathbf{N}} |\lambda a_n| = \sum_{n \in \mathbf{N}} |\lambda| |a_n| = |\lambda| \sum_{n \in \mathbf{N}} |a_n| = |\lambda| \|a\|_1$ .

“(N4)” Once again, for two sequences  $a$  and  $b$  of real numbers we have  $|a_n + b_n| \leq |a_n| + |b_n|$  for every  $n$ . This implies that for absolutely convergent sequences  $\|a + b\|_1 = \sum_{n \in \mathbf{N}} |a_n + b_n| \leq \sum_{n \in \mathbf{N}} (|a_n| + |b_n|) = \sum_{n \in \mathbf{N}} |a_n| + \sum_{n \in \mathbf{N}} |b_n| = \|a\|_1 + \|b\|_1$ . ■

### Exercise 7.

Show that for any norm  $\|\cdot\|$  we have  $\|x \pm y\| \geq \left| \|x\| - \|y\| \right|$ .

*Solution.* This is equivalent to

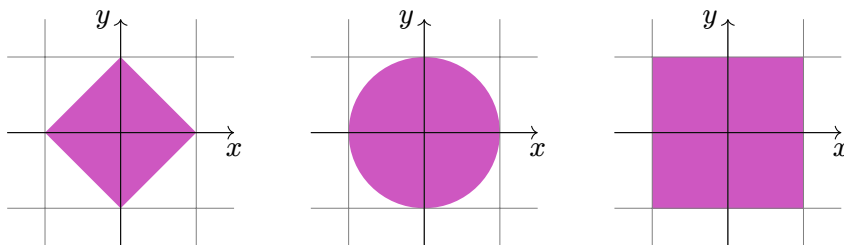
$$-\|x \pm y\| \leq \|x\| - \|y\| \leq \|x \pm y\|.$$

The right inequality is equivalent to  $\|x\| - \|y\| \leq \|x \pm y\|$ . But this follows from the triangle inequality applied to  $x = (x \pm y) \mp y$  and the fact that  $\| -y \| = \|y\|$ . The proof of the left inequality is similar. ■

**Exercise 8.**

Sketch open balls of radius 1 around the origin in  $\mathbf{R}^2$  with the different metrics  $d_1$ ,  $d_2$  and  $d_\infty$ .

*Solution.*



For  $d_1$ , the ball consist of all points  $(x, y)$  such that  $|x + y| < 1$ . The equation  $|x + y| = 1$  gives us 4 straight segments (depending on the positivity of  $x$  and  $y$ ) between the points  $(0, 1)$ ,  $(1, 0)$ ,  $(0, -1)$  and  $(-1, 0)$ . The open ball is the interior of the square delimited by these segments.

For  $d_2$ , the equation  $1 = d((x, y), 0) = \sqrt{x^2 + y^2}$  defines a circle of radius 1.

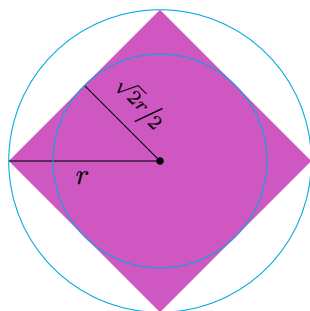
For  $d_\infty$ , the equation  $1 = d((x, y), 0) = \max(x, y)$  defines a square with corners  $(\pm 1, \pm 1)$ .

**Exercise 9.**

Let  $B_r(x)$  and  $B'_r(x)$  denote open balls of radius  $r$  in  $\mathbf{R}^2$  with respect to the Euclidean metric and the taxicab metric respectively. Find real numbers  $s$  and  $t$  such that

$$B_1(x) \subseteq B'_s(x) \subseteq B_{\sqrt{2}}(x) \subseteq B'_t(x) \subseteq B_2(x).$$

*Solution.* Each open ball  $B'_r(x)$  has bounding line of greatest Euclidean distance  $r$  from  $x$  and least distance  $r\sqrt{2}/2 = r/\sqrt{2}$ .



Therefore,

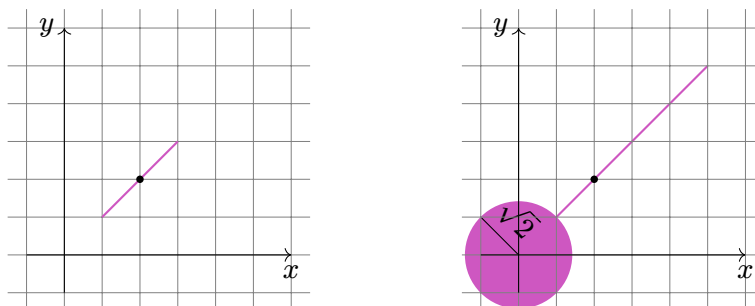
$$1 \leq \frac{s}{\sqrt{2}} < s \leq \sqrt{2} \leq \frac{t}{\sqrt{2}} < t \leq 2.$$

So the only possible solutions are  $s = \sqrt{2}$  and  $t = 2$ .

**Exercise 10.**

Sketch open balls of radius  $\sqrt{2}$  and  $3\sqrt{2}$  around the point  $(2, 2)$  in  $\mathbf{R}^2$  with respect to the British Railway metric (see Tutorial 1, Question 5).

*Solution.*



If the radius  $r$  of the ball is smaller than the Euclidean distance from the center  $c$  to  $0$ , then the balls consists of points  $p$  that are linearly dependent with  $c$  and of distance at most  $r$  to  $c$ . This is simply a line segment (on the line between  $0$  and  $c$ ) of length  $2r$  centred at  $c$ .

If  $r \geq d_2(c, 0)$ , then to the above line segment we can add  $B_{r-d_2(c,0)}(0; d_2)$ . Indeed, once we have reach  $0$  we can go back in any direction, but not further away than  $r - d_2(c, 0)$ .

**Exercise 11.**

Prove that if  $x, y \in (X, d)$  are points in a metric space with  $d(x, y) = 2r > 0$ , then  $B_r(x)$  and  $B_r(y)$  are disjoint. Is it possible that in a metric space  $X$  containing more than one point that the only open sets are  $\emptyset$  and  $X$ ?

*Solution.* Since  $d(x, y) > 0$  we know that  $x \neq y$ . Suppose that  $z \in B_r(x) \cap B_r(y)$ . Then  $d(x, y) \leq d(x, z) + d(z, y) < 2r$ , which contradicts  $d(x, y) = 2r$ .

It is not possible since if  $x, y \in X$  with  $x \neq y$ , then  $d(x, y) = 2r > 0$ , for some  $r \in \mathbf{R}$ . Then we know that  $B_r(x) \cap B_r(y) = \emptyset$  and both of these open balls are non-empty and open in  $X$ .

**To go further**

The same idea shows that if  $X$  is a metric space containing at least  $n$  points, then it has at least  $n$  open sets different from  $\emptyset$  and  $X$ .

**Exercise 1.**

Given any two subsets  $U, V \subseteq (X, d)$  of a metric space show that

- $U \subseteq V \implies U^\circ \subseteq V^\circ$ ;
- $(U^\circ)^\circ = U^\circ$ ;
- $U^\circ$  is open in  $X$ ;
- $U^\circ$  is the largest subset of  $U$  that is open in  $X$ .

*Solution.* a. If  $u \in U^\circ$ , then  $\exists \varepsilon > 0$  such that  $B_\varepsilon(u) \subseteq U \subseteq V$ ; so  $u \in V^\circ$ ;

b. By definition  $(U^\circ)^\circ \subseteq U^\circ$ . On the other hand, if  $u \in U^\circ$ , then  $\exists \varepsilon > 0$  such that  $B_\varepsilon(u) \subseteq U$ , so  $B_\varepsilon(u)^\circ \subseteq U^\circ$  by part a, but  $B_\varepsilon(u)$  is open by Lemma 1.5.12 and so  $B_\varepsilon(u) = B_\varepsilon(u)^\circ$ . Therefore,  $u \in (U^\circ)^\circ$ , so  $U^\circ \subseteq (U^\circ)^\circ$ .

c. This follows immediately from part b.

d. Suppose that  $U^\circ \subseteq W \subseteq U$ , where  $W$  is open in  $X$ . Then  $W = W^\circ \subseteq U^\circ$  by part a. So  $W \subseteq U^\circ$ , and  $U^\circ = W$ . ■

**Exercise 2.**

Show that an open subset  $U$  of a metric space is a union of open balls.

*Solution.* Since  $U$  is open, for every  $u \in U$ ,  $\exists \varepsilon_u > 0$  such that  $B_{\varepsilon_u}(u) \subseteq U$ . Therefore,  $\bigcup_{u \in U} B_{\varepsilon_u}(u) \subseteq U$ . And since  $u \in B_{\varepsilon_u}(u)$  we also have  $U \subseteq \bigcup_{u \in U} B_{\varepsilon_u}(u)$ . ■

**Exercise 3.**

Consider the set of binary sequences  $\{0, 1\}^\infty$  with the metric  $d^*$  from Tutorial 2, Question 3. What is the maximum possible value of  $d^*(x, y)$ , and for which pairs of sequences is this attained?

Describe the elements of the open ball  $B_{3/4}(0)$  in  $\{0, 1\}^\infty$ , where 0 denotes the zero sequence  $00\dots$

*Solution.* The maximum possible value of  $d^*(x, y)$  is  $\sum_{j=0}^{\infty} 1/2^j = 2$ . It is attained by any pair of sequences which differ in every element; for example  $d^*(0, 11\dots) = 2$ .

By definition,  $B_{3/4}(0)$  contains all  $x$  such that  $\sum_{j=0}^{\infty} 1/2^j < 3/4$ . Thus  $x_0 = 0$ , otherwise  $d^*(x, 0) \geq 1$ . If  $x_0 = 0$  and  $x_1 = 1$ , then  $d^*(x, 0) = 1/2 + \sum_{j=2}^{\infty} 1/2^j$ , so it is necessary that  $x_2 = 0$  and  $x_3 x_4 \dots \neq 11\dots$ . On the other hand, if  $x_0 = x_1 = 0$ , then  $d^*(x, 0) \leq 1/2$ .

So  $x \in B_{3/4}(0)$  if and only if  $x = 00z$  for any sequence  $z$ , or  $x = 010w$  for any sequence  $w \neq 11\dots$ . The additional elements in  $\overline{B}_{3/4}(0)$  are  $01011\dots$  and  $01100\dots$ . ■

**Exercise 4.**

On  $\mathbf{R}^n$  check that

$$d_\infty(x, y) \leq d_2(x, y) \leq d_1(x, y) \leq nd_\infty(x, y).$$

where  $d_1$  is the taxicab metric,  $d_2$  is the Euclidean metric, and  $d_\infty$  is the max metric.

*Solution.* The first and last inequalities follow from the definitions; you may find them easier to deduce by first considering the following observation:

$$(d_\infty(x, y))^2 \leq (d_2(x, y))^2 \leq (d_1(x, y))^2.$$

The final inequality follows directly from the definitions. ■

**Exercise 5.**

Let  $(X, d_X)$  and  $(Y, d_Y)$  be two metric spaces and consider one of the metrics  $d_a, d_b, d_c$  on  $X \times Y$ .

- a. Prove that the projection maps:

$$p_1: X \times Y \rightarrow X \quad \text{and} \quad p_2: X \times Y \rightarrow Y$$

given by  $p_1(x, y) = x$  and  $p_2(x, y) = y$  are continuous.

- b. For some fixed  $x_0 \in X$  and  $y_0 \in Y$  prove that

$$i_{y_0}: X \rightarrow X \times Y \quad \text{and} \quad i_{x_0}: Y \rightarrow X \times Y$$

given by  $i_{y_0}(x) = (x, y_0)$  and  $i_{x_0}(y) = (x_0, y)$  are continuous.

*Solution.* Since  $d_a, d_b$  and  $d_c$  have the same open sets, continuity for  $d_a$  is equivalent to continuity for  $d_b$ , and also to continuity for  $d_c$ .

- a. We will consider the metric  $d_a$  on  $X \times Y$  and prove that  $p_1$  is continuous. Let  $(x, y) \in X \times Y$  and  $\varepsilon > 0$ . Then

$$d_X(p_1(x, y), p_1(x', y')) = d_X(x, x') \leq d_a((x, y), (x', y')) = d_X(x, x') + d_Y(y, y').$$

So we find that  $p_1$  is continuous at  $(x, y) \in X \times Y$  by taking  $\delta = \varepsilon$ . The proof for  $p_2$  is symmetric.

- b. We will prove that  $i_{y_0}$  is continuous taking the metric  $d_a$  again on  $X \times Y$ . Let  $x \in X$  and  $\varepsilon > 0$ . Then

$$d_a(i_{y_0}(x), i_{y_0}(x')) = d_a((x, y_0), (x', y_0)) = d_X(x, x') + d_Y(y_0, y_0) = d_X(x, x').$$

So by taking  $\delta = \varepsilon$  again we see that  $i_{y_0}$  is continuous at  $x \in X$ . The proof for  $p_2$  is symmetric. ■

**Exercise 6.**

Given a metric space  $(X, d)$ , consider  $(X \times X, d_a)$ .

- a. Prove that the diagonal map

$$\Delta: X \rightarrow X \times X$$

given by  $\Delta(x) = (x, x)$  is continuous.

- b. Show that the metric  $d: (X \times X, d_a) \rightarrow (\mathbf{R}, d_{\mathbf{R}})$  is a continuous function.

*Solution.* a. Let  $x \in X$  and  $\varepsilon > 0$ . Then

$$d_c(\Delta(x), \Delta(y)) = d_a((x, x), (y, y)) = \max\{d(x, y), d(x, y)\} = d(x, y).$$

So we can take  $\delta = \varepsilon$ .

- b. Let  $(x, y) \in X \times X$  and  $\varepsilon > 0$ . Then we need to find  $\delta > 0$  such that

$$d_a((x, y), (x', y')) < \delta \implies |d(x, y) - d(x', y')| < \varepsilon.$$

We have

$$\begin{aligned} |d(x, y) - d(x', y')| &\leq |(d(x, x') + d(x', y') + d(y', y)) - d(x', y')| \\ &= d(x, x') + d(y, y'). \end{aligned}$$

Since  $d_a((x, y), (x', y')) = d(x, x') + d(y, y')$  we can take  $\delta = \varepsilon$ . ■

**Exercise 7.**

Suppose  $(X, d)$  is a metric space. Define  $d'(x, y) = \min\{1, d(x, y)\}$ . Prove that  $d'$  is a metric on  $X$  and then show that  $d$  and  $d'$  are topologically equivalent. Give an example to show that they may not be Lipschitz equivalent.

*Solution.* We have  $d'(x, y) \leq d(x, y)$  and  $0 < d'(x, y) \leq 1$ , for all  $x \neq y \in X$ . So for the triangle inequality notice that

$$\begin{aligned} d'(x, z) &= \min\{1, d(x, z)\} \leq \min\{1, d(x, y) + d(y, z)\} \\ &\leq \min\{1, d(x, y)\} + \min\{1, d(y, z)\} = d'(x, y) + d'(y, z). \end{aligned}$$

The other axioms of a metric are straightforward to check.

In terms of open balls, notice that

$$B_r(x; d') = \begin{cases} B_r(x; d), & \text{if } r \leq 1; \\ X, & \text{if } r > 1, \end{cases}$$

so  $B_r(x; d) \subseteq B_r(x; d')$ , for all  $x \in X$  and  $r > 0$ . Therefore, it automatically follows that if  $U \subseteq X$  is  $d'$ -open, then it is  $d$ -open. Suppose now that  $U \subseteq X$  is  $d$ -open and

let  $u \in U$ . Then  $\exists \varepsilon > 0$  such that  $B_\varepsilon(u; d) \subseteq U$ . Let  $\varepsilon' := \min\{1, \varepsilon\} \leq 1$ . Then  $B_{\varepsilon'}(u; d') \subseteq B_\varepsilon(u; d) \subseteq U$  so  $U$  is  $d'$ -open.

Obviously, for any metric space  $(X, d)$ , we find that  $(X, d')$  is bounded. But an unbounded metric space cannot be Lipschitz equivalent to a bounded one by Lemma 1.9.4. For a concrete counterexample, take  $X = \mathbf{R}$  and  $d = d_{\mathbf{R}}$  its standard metric. If  $d$  and  $d'$  were Lipschitz equivalent, then there would exist  $h, k \in (0, \infty)$  such that

$$hd'(x, y) \leq d(x, y) = |x - y| \leq kd'(x, y) \leq k, \quad \forall x, y \in \mathbf{R}.$$

Taking  $x$  and  $y$  to be further than distance  $k$  away from each other, for example,  $x = 0, y = k + 1$ , we find a contradiction. ■

### Exercise 8.

Prove that isometries and Lipschitz equivalences between metric spaces are continuous.

*Solution.* Isometries are continuous by taking  $\delta = \varepsilon$  in the definition of continuity. A Lipschitz equivalence  $f: (X, d_X) \rightarrow (Y, d_Y)$  (using the same notation as in its definition) is continuous at  $x \in X$  by taking  $\delta = h\varepsilon$  since:

$$d_Y(f(x), f(y)) \leq \frac{1}{h}d_X(x, y) < \frac{\delta}{h} = \varepsilon. \quad \blacksquare$$

### Exercise 9.

Check that being an isometry defines an equivalence relation on all metric spaces. Do the same for Lipschitz equivalences and homeomorphisms.

*Solution.* “ $X \sim X$ ” The identity map  $\text{Id}: X \rightarrow X$  is bijective and an isometry as well as being its own inverse. It is thus a Lipschitz equivalence and a continuous map.

“ $X \sim Y \Rightarrow Y \sim X$ ” If  $f: X \rightarrow Y$  is an isometry, respectively a homeomorphism, then  $f^{-1}: Y \rightarrow X$  is automatically an isometry, respectively a homeomorphism. If  $f$  is a Lipschitz equivalence, with parameters  $h$  and  $k$ , then

$$hd_Y(f(x), f(y)) \leq d_X(x, y) \leq kd_Y(f(x), f(y)).$$

By letting  $w = f(x)$  and  $z = f(y)$  we have

$$hd_Y(w, z) \leq d_X(f^{-1}(w), f^{-1}(z)) \leq kd_Y(w, z),$$

which is equivalent to

$$\frac{1}{k}d_X(f^{-1}(w), f^{-1}(z)) \leq d_Y(w, z) \leq \frac{1}{h}d_Y(w, z).$$

So  $f^{-1}$  is also a Lipschitz equivalence (with parameters  $1/k$  and  $1/h$ ).

“ $X \sim Y, Y \sim Z \Rightarrow X \sim Z$ ” This follows directly from the composition of continuous functions. ■

**Exercise 10.**

In the proof that two Lipschitz equivalent metrics  $d$  and  $d'$  on  $X$  are topologically equivalent (Proposition 1.7.6) prove the statements regarding open balls.

That is, show that if  $f$  is a  $(h, k)$ -Lipschitz equivalence, then  $f(B_{h\varepsilon}(x)) \subseteq B_\varepsilon(f(x))$  and  $f^{-1}(B_{\frac{\varepsilon}{k}}(y)) \subseteq B_\varepsilon(f^{-1}(y))$ .

*Solution.* Let  $y \in B_{h\varepsilon}(x; d)$ . Then  $d(x, y) < h\varepsilon$ , which implies that  $d'(f(x), f(y)) \leq \frac{1}{h}d(x, y) < \varepsilon$ . This proves that  $f(y) \in B_\varepsilon(f(x))$  for any  $y \in B_{h\varepsilon}(x)$ .

The formula for  $f^{-1}$  follows from the above formula and the fact that  $f^{-1}$  is a  $(1/k, 1/h)$ -Lipschitz equivalence. ■

**Exercise 1.**

Prove that the **open annulus**  $\{x \in \mathbf{R}^2 \mid 4 < x_1^2 + x_2^2 < 9\}$  and the **punctured plane**  $\mathbf{R}^2 \setminus \{0\}$  are homeomorphic subspaces of  $\mathbf{R}^2$ .

*Solution.* The annulus  $A := \{x \in \mathbf{R}^2 \mid 4 < x_1^2 + x_2^2 < 9\}$  lies between the circles of radius 2 and 3, so every  $a \in A$  may be written as  $(r \cos \theta, r \sin \theta)$  for some  $r \in (2, 3) \subseteq \mathbf{R}$ . By Example 1.7.5, there exists a homeomorphism  $t: (2, 3) \rightarrow (0, +\infty)$  (for example, one can take  $t(x) = \tan(x\pi/2)$ ). Define

$$h: A \rightarrow \mathbf{R}^2 \setminus \{0\} \quad \text{by} \quad h(a) = (t(r) \cos \theta, t(r) \sin \theta).$$

This is continuous by the composition of continuous functions and it has a continuous inverse  $h^{-1}(s \cos \theta, s \sin \theta) = (t^{-1}(s) \cos \theta, t^{-1}(s) \sin \theta)$ . For our concrete example, one has  $t^{-1}(s) = \frac{2}{\pi} \tan^{-1} s$ . ■

**Exercise 2.**

Let  $\ell^1 = (\mathcal{Y}, d_1)$  be the sequence space of absolutely convergent sequences (see Definition 1.4.3) with the  $d_1$  metric.

Find a distance preserving injection of the space  $(\{0, 1\}^\infty, d^*)$  (see Tutorial 3, Question 4) into  $\ell^1$ .

Find an example of a point  $a \in \mathcal{Y}$  such that  $a \in \overline{B}_{1/10}(0)$  but  $a \notin B_{1/10}(0)$ .

*Solution.* Consider the function  $i: \{0, 1\}^\infty \rightarrow \mathcal{Y}$  defined by

$$i(x_0 x_1 \dots x_k \dots) = (x_0, x_1/2, \dots, x_k/2^k, \dots);$$

the right hand side is well defined, because

$$\sum_{k \in \mathbf{N}} \left| \frac{x_k}{2^k} \right| \leq \sum_{k \in \mathbf{N}} \frac{1}{2^k} = 2,$$

as required. Obviously  $i$  is injective; it also preserves distance, because

$$d^*(x, y) = \sum_{k \in \mathbf{N}} \frac{|x_k - y_k|}{2^k} = \sum_{k \in \mathbf{N}} \left| \frac{x_k}{2^k} - \frac{y_k}{2^k} \right| = d_1(i(x), i(y)).$$

A point  $a \in \mathcal{Y}$  satisfies the condition if and only if  $\sum_{k=0}^{\infty} |a_k| = 1/10$ . Any example, such as

$$\left( \frac{1}{10}, 0, 0, \dots \right) \quad \text{or} \quad \left( \frac{1}{20}, \frac{1}{40}, \dots, \frac{1}{10 \cdot 2^{j+1}}, \dots \right)$$

will do. ■

**Exercise 3.**

Given any two subsets  $U, V \subseteq X$  prove the following:

- $U \subseteq V$  implies  $\bar{U} \subseteq \bar{V}$ ;
- $\overline{\bar{V}} = \bar{V}$ ;
- $\bar{V}$  is closed in  $X$ ;
- $\bar{V}$  is the smallest set containing  $V$  that is closed in  $X$ .

*Solution.* a. For every  $\varepsilon > 0$ , if  $B_\varepsilon(x) \cap U$  is non-empty, so is  $B_\varepsilon(x) \cap V$ .

b. By definition,  $\bar{V} \subseteq \overline{\bar{V}}$ . Conversely, if  $x \in \overline{\bar{V}}$ , then  $B_{\varepsilon/2}(x) \cap \bar{V}$  is non-empty for every  $\varepsilon > 0$ . So there exists  $w$  for which  $d(x, w) < \varepsilon/2$  and  $w \in \bar{V}$ , which implies that  $B_{\varepsilon/2}(w) \cap V$  is non-empty; therefore  $B_\varepsilon(x) \cap V$  is also non-empty, by the triangle inequality. Hence  $x \in \bar{V}$  and  $\overline{\bar{V}} \subseteq \bar{V}$  as required.

c. This follows immediately from b.

d. Suppose that  $V \subseteq W \subseteq \bar{V}$ , where  $W$  is closed. Then  $\bar{V} \subseteq \bar{W} = W$  by a. So  $\bar{V} \subseteq W$ , and  $W = \bar{V}$  as required. ■

**Exercise 4.**

Determine whether the following subsets are open, closed, or neither in  $\mathbf{R}$  (with the standard metric):

- $[0, 1)$ ;
- $\mathbf{Z}$ ;
- $\{x \in \mathbf{R} \mid \sin x > 0\}$ ;
- $\bigcup_{n=2}^{\infty} [1/n, 1)$ .

*Solution.* a.  $[0, 1)$  is not open since 0 does not lie in the interior. It is not closed because 1 is a closure point that does not belong to the set.

b.  $\mathbf{Z}$  is closed in  $\mathbf{R}$  because its complement is open as a union of open intervals.

c.  $\{x \in \mathbf{R} \mid \sin x > 0\} = \bigcup_{n \in \mathbf{Z}} (2n\pi, (2n+1)\pi)$  is a union of open intervals in  $\mathbf{R}$ , and is therefore open.

d.  $\bigcup_{n=2}^{\infty} [1/n, 1) = (0, 1)$  is open in  $\mathbf{R}$ . ■

**Exercise 5.**

For a subset  $W \subseteq (X, d)$  of a metric space prove that

- $W^\circ = X \setminus (\overline{X \setminus W})$ ;
- $\overline{W} = X \setminus (X \setminus W)^\circ$ .

*Solution.* a.  $w \in W^\circ \iff \exists \varepsilon > 0 : B_\varepsilon(w) \subseteq W \iff \exists \varepsilon > 0 : B_\varepsilon(w) \cap (X \setminus W) = \emptyset$   
 $\iff w \notin \overline{X \setminus W}$ .

b.  $x \in \overline{W} \iff \forall \varepsilon > 0 : B_\varepsilon(x) \cap W \neq \emptyset \iff \forall \varepsilon > 0 : B_\varepsilon(x) \not\subseteq X \setminus W$   
 $\iff x \notin (X \setminus W)^\circ$ . ■

**Exercise 6.**

In a metric space  $(X, d)$  prove that every open ball is contained in a closed ball and that every closed ball is contained in an open ball. Is it always true that  $\overline{B_r(x)}$  is  $\overline{B_r(x)}$ ? Give a proof or a counterexample.

*Solution.* For any  $x \in X$  and  $r > 0$ ,

$$B_r(x) \subseteq \overline{B_r(x)} \subseteq B_{2r}(x).$$

It is not true that the closure of  $B_r(x)$  is always  $\overline{B_r(x)}$ . In a discrete metric space  $(X, d)$  we the ball  $B_1(x) = \{x\}$  is both open and close, so  $\overline{B_1(x)} = \{x\}$  as well. But  $\overline{B_1(x)} = X$ , which is not the same as  $\{x\}$  if  $X$  contains more than one point. ■

**Exercise 7.**

Consider a subset  $W \subseteq (X, d)$  of a metric space. Recall that a point  $x \in X$  is an accumulation point of  $W$  if, for every  $\varepsilon > 0$ ,  $B_\varepsilon(x) \cap W \setminus \{x\} \neq \emptyset$ . Let  $\text{Acc}(W)$  denote the set of all accumulation points of  $W$ . Prove that  $\overline{W} = W \cup \text{Acc}(W)$ . Recall that  $w \in W$  is an isolated point of  $W$  if there exists  $\varepsilon > 0$  such that  $B_\varepsilon(w) \cap W = \{w\}$ . Let  $\text{Iso}(W)$  denote the set of all isolated points of  $W$ . Prove that  $\text{Iso}(W) = W \setminus \text{Acc}(W)$ .

*Solution.* Let  $x \in \overline{W}$ . Then if  $x \notin W$  we have have that  $B_\varepsilon(x) \cap W \setminus \{x\} \neq \emptyset$ , for every  $\varepsilon > 0$ . So  $x \in \text{Acc}(W)$ . It is easy to see that if  $x \in \text{Acc}(W)$ , then  $x \in \overline{W}$ .

Let  $w \in \text{Iso}(W)$ . Then  $w \in W$  by definition and  $B_\varepsilon(w) \cap W = \{w\}$ , for some  $\varepsilon > 0$ , which implies that  $B_\varepsilon(w) \cap W \setminus \{w\} = \emptyset$  so  $w \notin \text{Acc}(W)$ . Suppose that  $w \in W \setminus \text{Acc}(W)$ . Then there is an  $\varepsilon > 0$  such that  $B_\varepsilon(w) \cap W \setminus \{w\} = \emptyset$  which implies that  $B_\varepsilon(w) \cap W = \{w\}$  so  $w \in \text{Iso}(W)$ . ■

**Exercise 1.**

Prove that if  $Y \subseteq Z$ , where  $Z$  is a bounded subset of  $(X, d)$ , then  $Y$  is bounded and  $\text{diam } Y \leq \text{diam } Z$ .

*Solution.* Since  $Z$  is bounded we know there exists  $x \in X$  and  $r > 0$  such that  $Z \subseteq \overline{B}_r(x)$  and since  $Y \subseteq Z$  we have that  $Y$  is bounded. Also, for each  $y_1, y_2 \in Y \subseteq Z$ ,  $d(y_1, y_2) \leq \text{diam } Z$  so  $\text{diam } Z$  is an upper bound for  $\{d(y_1, y_2) \mid y_1, y_2 \in Y\}$  which implies that  $\text{diam } Y := \sup\{d(y_1, y_2) \mid y_1, y_2 \in Y\} \leq \text{diam } Z$ . ■

**Exercise 2.**

Prove that if  $Y, Z$  are bounded subsets of  $(X, d)$  with  $Y \cap Z \neq \emptyset$ , then

$$\text{diam } Y \cup Z \leq \text{diam } Y + \text{diam } Z.$$

*Solution.* If  $u_1, u_2 \in Y \cup Z$  both belong to  $Y$ , then

$$d(u_1, u_2) \leq \text{diam } Y \leq \text{diam } Y + \text{diam } Z.$$

If  $u_1, u_2$  both belong to  $Z$ , a interchanging the roles of  $Y$  and  $Z$  gives the desired result. If  $u_1 \in Y$  and  $u_2 \in Z$  (or the the other way around) then take some  $\alpha \in Y \cap Z$  and

$$d(u_1, u_2) \leq d(u_1, \alpha) + d(\alpha, u_2) \leq \text{diam } Y + \text{diam } Z.$$

Therefore,  $d(u_1, u_2) \leq \text{diam } Y + \text{diam } Z$ , for all  $u_1, u_2 \in Y \cup Z$ , which gives the result. ■

**Exercise 3.**

For a subset  $W \subseteq (X, d)$  of a metric space, prove that  $\partial W = \overline{W} \cap \overline{(X \setminus W)}$ .

*Solution.*  $x \in \partial W \iff x \in \overline{W} \setminus W^\circ \iff (x \in \overline{W} \& \forall \varepsilon > 0 : B_\varepsilon(x) \cap (X \setminus W) \neq \emptyset) \iff (x \in \overline{W} \& x \in \overline{(X \setminus W)})$ . ■

**Exercise 4.**

Determine  $\partial S$ ,  $S^\circ$ , and  $\overline{S}$  for each of the subspaces  $S := [0, 1) \cup (2, 3]$ ,  $S := \mathbf{Z}$ , and  $S := \mathbf{R} \setminus \mathbf{Q}$  of the Euclidean line.

*Solution.*

$$\begin{aligned} \partial[0, 1) \cup (2, 3] &= \{0, 1, 2, 3\}, & ([0, 1) \cup (2, 3])^\circ &= (0, 1) \cup (2, 3), \\ \overline{[0, 1) \cup (2, 3]} &= [0, 1] \cup [2, 3]. \end{aligned}$$

$$\partial\mathbf{Z} = \mathbf{Z}, \quad (\mathbf{Z})^\circ = \emptyset, \quad \overline{\mathbf{Z}} = \mathbf{Z}.$$

$$\partial(\mathbf{R} \setminus \mathbf{Q}) = \mathbf{R}, \quad (\mathbf{R} \setminus \mathbf{Q})^\circ = \emptyset, \quad \overline{(\mathbf{R} \setminus \mathbf{Q})} = \mathbf{R}. \quad \blacksquare$$

**Exercise 5.**

Is  $\mathbf{Q}^n$  dense in  $\mathbf{R}^n$ ? Let  $\mathbf{Z}[\frac{1}{2}] := \{\frac{m}{2^n} \mid m, n \in \mathbf{Z}\}$ . Is  $\mathbf{Z}[\frac{1}{2}]$  dense in  $\mathbf{R}$ ? Is  $\mathbf{Z}[\frac{1}{2}]^n$  dense in  $\mathbf{R}^n$ .

*Solution.* To prove that  $\mathbf{Q}^n$  is dense in  $\mathbf{R}^n$  we must prove that  $\overline{\mathbf{Q}^n} = \mathbf{R}^n$ . It is sufficient to show that every  $(x_1, \dots, x_n) \in \mathbf{R}^n$  is the limit of a sequence in  $\mathbf{Q}^n$ . But we know that  $\mathbf{Q}$  is dense in  $\mathbf{R}$ , so there exists a sequence  $(q_{i,k})_{k \geq 1}$  converging to  $x_i$  for every  $1 \leq i \leq n$ . Then  $(q_k = (q_{1,k}, \dots, q_{n,k}))_{k \geq 1}$  converges to  $x$ , because

$$d(x, q_k) = \left( \sum_{i=1}^n (x_i - q_{i,k})^2 \right)^{1/2} \rightarrow 0, \quad \text{as } k \rightarrow \infty.$$

Given any  $z \in \mathbf{R}$ , let  $z_n$  denote the truncation of its binary expansion after  $n$  digits, for every  $n \geq 1$ . So  $z_n \in \mathbf{Z}[\frac{1}{2}]$  by construction, and  $z_n \rightarrow z$  as  $n \rightarrow \infty$ . So  $\mathbf{Z}[\frac{1}{2}]$  is dense in  $\mathbf{R}$ . Using a similar argument as for  $\mathbf{Q}^n$  shows that  $\mathbf{Z}[\frac{1}{2}]^n$  is dense in  $\mathbf{R}^n$ . ■

**Exercise 6.**

Show that a subset  $U \subseteq X$  of a metric space is open if and only if for every sequence  $(x_n)$  of  $X$  that converges to a point in  $U$ , there exists  $K \in \mathbf{N}$  such that  $x_k \in U$ , for all  $k \geq K$ .

*Solution.* “ $\Rightarrow$ ” Suppose that  $U$  is open in  $X$  and that  $(x_n)$  is a sequence in  $X$  that converges to  $u \in U$ . Since  $U$  is open, there exists  $\varepsilon > 0$  such that  $B_\varepsilon(u) \subseteq U$ , and for this  $\varepsilon > 0$ ,  $\exists K \in \mathbf{N}$  such that  $k \geq K \implies d(x_k, u) < \varepsilon$ , i.e.  $x_k \in B_\varepsilon(u) \subseteq U$ , for all  $k \geq K$ .

“ $\Leftarrow$ ” Let  $u \in U$  and suppose  $u \notin U^\circ$ , that is, for every  $\varepsilon > 0$ ,  $B_\varepsilon(u) \cap (X \setminus U) \neq \emptyset$ . Choose  $x_i \in B_{1/i}(u) \cap (X \setminus U)$ . Then  $(x_n)$  is a sequence in  $X$  such that  $\lim_n x_n = u$  but  $x_k \notin U$ , for all  $k \in \mathbf{N}$ . ■

**Exercise 7.**

Let  $\mathcal{P}[0, 1] \subseteq \mathcal{C}[0, 1]$  be the subspace consisting of all *polynomials functions*. By constructing the sequence

$$f_n(x) := 1 + \frac{x}{2} + \frac{x^2}{4} + \dots + \frac{x^n}{2^n} \in \mathcal{P}[0, 1],$$

show that  $\mathcal{P}[0, 1]$  is not closed in  $\mathcal{C}[0, 1]$ .

*Solution.* Let  $f(x) = (1 - x/2)^{-1}$  in  $\mathcal{C}[0, 1]$ ; then the Binomial Theorem implies that  $f(x) - f_n(x) = \sum_{k \geq n+1} x^k/2^k \leq 1/2^n$  for  $x \in [0, 1]$ . So

$$d_{\text{sup}}(f_n, f) = \sup_{x \in [0, 1]} |f_n(x) - f(x)| = 1/2^n \rightarrow 0, \quad \text{as } n \rightarrow \infty,$$

and  $f_n \rightarrow f$  in  $\mathcal{C}[0, 1]$ . But  $f$  is not a polynomial, so  $\mathcal{P}[0, 1]$  is not closed. ■

**Exercise 8.**

Given the sequence space  $(\{0, 1\}^\infty, d^*)$  (see Tutorial 3 Question 4), consider  $f_1: \{0, 1\}^\infty \rightarrow \{0, 1\}^\infty$  defined by inserting 1 at the beginning of every sequence. Prove that  $f_1$  is continuous and decreases the distance in the sense that  $d^*(f_1(x), f_1(y)) \leq d^*(x, y)$ .

*Solution.* By definition  $f_1(x) = 1x_0x_1 \dots$  and  $f_1(y) = 1y_0y_1 \dots$ . So

$$d^*(f_1(x), f_1(y)) = \sum_{j=1}^{\infty} \frac{|x_{j-1} - y_{j-1}|}{2^j} = \frac{1}{2} \sum_{j=1}^{\infty} \frac{|x_{j-1} - y_{j-1}|}{2^{j-1}} = \frac{1}{2} d^*(x, y).$$

In particular,  $d^*(f_1(x), f_1(y)) \leq d^*(x, y)$ , with equality only if  $x = y$ .

It also follows that  $B_{2\varepsilon}(x) = B_\varepsilon(f_1(x))$  for any  $\varepsilon > 0$ , and therefore that  $f_1$  is continuous by choosing  $\delta \leq 2\varepsilon$ . ■

**Exercise 9.**

Show that if  $A \subseteq \mathbf{R}^n$  is a bounded subset, then  $A \subseteq [a, b]^n$  for some  $a, b \in \mathbf{R}$ .

*Solution.* Since  $A$  is bounded, it is contained in a closed ball  $\overline{B}_r(x)$  for some  $x = (x_1, \dots, x_n) \in \mathbf{R}^n$ . Let  $a := \min\{x_i - r \mid i \in \{1, \dots, n\}\}$  and  $b := \max\{x_i + r \mid i \in \{1, \dots, n\}\}$ . Then  $A \subseteq \overline{B}_r(x) \subseteq [a, b]^n$  as desired. ■

Exercises marked with a \* are slightly harder bonus questions.

**Exercise 1.**

Which of the following subsets of  $\mathbf{R}^2$  are connected? Which are path-connected?

- $B_1((1,0)) \cup B_1((-1,0))$ ;
- $\overline{B_1}((1,0)) \cup \overline{B_1}((-1,0))$ ;
- $\overline{B_1}((1,0)) \cup B_1((-1,0))$ ;
- The “rational comb”  $C := \{(q, y) \in \mathbf{R}^2 \mid q \in \mathbf{Q}, y \in [0, 1]\} \cup (\mathbf{R} \times \{1\})$ ;
- The set of all points in  $\mathbf{R}^2$  with at least one coordinate in  $\mathbf{Q}$ .

- Solution.*
- This is a disjoint union of two open sets. It is thus not connected and therefore not path-connected.
  - This is a union of two path-connected sets (the two balls) with non-empty intersection (it contains  $(0,0)$ ). It is therefore path-connected, and hence connected.
  - The two balls are path-connected. For  $x$  in  $B_1((-1,0))$  the line segment between  $0$  and  $x$  stays inside  $\overline{B_1}((1,0)) \cup B_1((-1,0))$  and is thus a path from  $0$  to  $x$ . We conclude that  $\overline{B_1}((1,0)) \cup B_1((-1,0))$  is path-connected, and hence connected.
  - A point of the form  $(q, y)$  ( $q \in \mathbf{Q}, y \in [0, 1]$ ) is connected by a (vertical) path to  $(q, 1)$ , which is connected by a (horizontal path) to  $(0, 1)$ . All points of the form  $(r, 1)$  are connected by a path to  $(0, 1)$ . We conclude that  $C$  is path-connected, and hence connected.
  - Any point  $(p, r)$  with  $p \in \mathbf{Q}$  is connected by a vertical path to  $(p, 0)$ , which is connected to  $(0, 0)$  by a horizontal path. Similarly, any point  $(r, q)$  with  $q \in \mathbf{Q}$  is path-connected. We conclude that our set is path-connected, and hence connected. ■

**Exercise 2.**

Suppose that  $A \mid B$  is a partition of a metric space  $X$  and that  $f: X \rightarrow Y$  is a function to another metric space  $Y$ . Prove that if the restrictions  $f|_A$  and  $f|_B$  are both continuous, then  $f$  is continuous.

*Solution.* Let  $U$  be an open subset of  $Y$ . Set  $U_A := f^{-1}(U) \cap A$  and  $U_B := f^{-1}(U) \cap B$ . Since both  $f|_A$  and  $f|_B$  are continuous we find that  $U_A = (f|_A)^{-1}(U)$  is open in  $A$  and  $U_B = (f|_B)^{-1}(U)$  is open in  $B$ . Since  $A \mid B$  is a partition of  $X$  both  $U_A$  and  $U_B$  are also open in  $X$ . Therefore,  $f^{-1}(U) = U_A \cup U_B$  is open in  $X$ . ■

**Exercise 3.**

Suppose that for each  $i \in \{1, \dots, n\}$  that  $A_i$  is a connected subspace of a metric space  $X$ , such that  $A_i \cap A_{i+1} \neq \emptyset$  for each  $i \in \{1, \dots, n-1\}$ . Prove that  $\bigcup_{i=1}^n A_i$  is connected. Does this result extend to an infinite sequence  $(A_i)$  of connected subsets?

*Solution.* Suppose  $f: \bigcup_{i=1}^n A_i \rightarrow \{0, 1\}$  is continuous. The restriction  $f|_{A_i}: A_i \rightarrow \{0, 1\}$  is continuous and therefore constant since  $A_i$  is connected. Let us write  $c_i$  for the value  $f|_{A_i}(x)$  (for any  $x \in A_i$ ). Since the intersection  $A_i \cap A_{i+1}$  is not empty, then  $c_i = c_{i+1}$  for all  $i$ . We conclude that all the  $c_i$  are equal and thus that  $f$  is constant.

The infinite case is also true, with the same proof. ■

**Exercise 4.**

Show that a metric space  $X$  is connected if and only if for every  $W \subseteq X$  such that  $W \neq \emptyset, X$ , the boundary  $\partial W$  is non-empty.

*Solution.* A metric space  $X$  is connected if and only if the only subsets which are both open and closed are  $\emptyset$  and  $X$ . But  $W$  is both open and closed if and only if  $\overline{W} = W^\circ$ , if and only if  $\partial W = \emptyset$ . ■

**Exercise 5.**

If  $X$  is a connected metric space containing more than one point, and if  $\{x\}$  is closed for every  $x \in X$ , show that  $X$  is infinite.

*Solution.* If  $X$  is finite and all points are closed, then  $X \setminus \{x\}$  is a finite union of closed sets and is therefore closed. Therefore,  $\{x\} = X \setminus (X \setminus \{x\})$  is both open and closed and  $X$  cannot be connected (unless  $X = \{x\}$ ). ■

**Exercise 6.**

Prove that any continuous function  $f: [a, b] \rightarrow [a, b]$  has a fixed point, that is, a point  $x$  such that  $f(x) = x$ .

*Hint: Consider the function  $g: [a, b] \rightarrow \mathbf{R}$  given by  $g(x) = f(x) - x$ .*

*Solution.* The function  $g$  in the hint is a continuous function from a closed interval to  $\mathbf{R}$  and, therefore, its image is a closed interval. Since  $a \leq f(x) \leq b$ , for all  $x \in [a, b]$ , we have

$$0 \leq g(a) = f(a) - a \leq b - a \quad \text{and} \quad a - b \leq g(b) = f(b) - b \leq 0.$$

So if  $f(a) \neq a$ , then  $g(a) > 0$ , and if  $f(b) \neq b$ , then  $g(b) < 0$ . So by the Intermediate Value Theorem, there exists  $x \in (a, b)$  such that  $g(x) = 0$ , i.e.  $f(x) = x$ . ■

**Exercise 7.**

- a. Prove that if  $X$  is a connected metric space and there exists a non-constant continuous function  $f: X \rightarrow \mathbf{R}$ , then  $X$  is uncountable.
- b. Prove that a connected metric space containing more than one point contains uncountably many points.

*Solution.* a. If  $f$  takes two real values  $a < b$ , then by the Intermediate Value Theorem, it takes on all values in  $[a, b]$ , which is uncountable, and hence so is  $X$ .

- b. Apply part a to  $f: X \rightarrow \mathbf{R}$  given by  $f(x) = d(x, a)$  for a fixed  $a \in X$ . ■

Exercises marked with a \* are slightly harder bonus questions.

**Exercise 1.**

Show that there is no continuous injective function  $f: \mathbf{R}^2 \rightarrow \mathbf{R}$ .

*Hint: Consider a triangle in  $\mathbf{R}^2$ .*

*Solution.* Consider a triangle  $ABC$  in  $\mathbf{R}^2$ . Then  $BC$  and  $BA \cup AC$  are both connected. So both  $f(BC)$  and  $f(BA \cup AC)$  contain  $[f(B), f(C)]$  by the Intermediate Value Theorem. ■

**\*Exercise 2.**

Suppose that  $f: [a, b] \rightarrow \mathbf{R}$  is a function such that  $|f^{-1}(y)| = 2$ , for every  $y$  in the image of  $f$ . Prove that  $f$  cannot be continuous.

*Solution.* For every  $x \in [a, b]$  there exists a unique  $x' \in [a, b]$ , with  $x \neq x'$ , such that  $f(x) = f(x')$ . We define a function that switches these points:

$$g: [a, b] \longrightarrow [a, b]$$

$$x \longmapsto g(x) = f^{-1}(f(x)) \setminus \{x\}.$$

If  $f$  is continuous,  $x \in [a, b]$ , and  $(x_n)$  is a sequence in  $[a, b]$  such that  $\lim x_n = x$ , then

$$\lim g(x_n) = \lim(f^{-1}(f(x_n)) \setminus \{x_n\}) = g(x)$$

since  $\lim f(x_n) = f(x)$ . So  $g$  is continuous but has no fixed point, contradicting Question 6. ■

**Exercise 3.**

- Suppose that  $f: X \rightarrow Y$  is a continuous function between metric spaces and that  $X$  is path-connected. Show that  $f(X)$  is path-connected.
- Suppose that  $X$  and  $Y$  are homeomorphic metric spaces. Show that  $X$  is path-connected if and only if  $Y$  is path-connected.
- Show that the graph  $G_f$  of a continuous function with path-connected domain is path-connected.

*Solution.* a. Let  $y_1, y_2 \in f(X)$ . Then there exists  $x_1, x_2 \in X$  such that  $y_1 = f(x_1)$  and  $y_2 = f(x_2)$ . Since  $X$  is path-connected, there exists a path  $g: I \rightarrow X$  from  $x_1$  to  $x_2$ . The composition  $f \circ g: I \rightarrow Y$  is continuous, as a composition of continuous functions, and therefore gives a path in  $Y$  from  $y_1$  to  $y_2$ .

- Suppose that  $f: X \rightarrow Y$  is a homeomorphism. By the first part, if  $X$  is path-connected then  $f(X) = Y$  is path-connected. Similarly, if  $Y$  is path-connected then  $X = f^{-1}(Y)$  is path-connected.

- c. Recall that the graph  $G_f$  of a continuous function is homeomorphic to its domain. Therefore, if the domain is path-connected, then  $G_f$  is path-connected by the first part. ■

**\*Exercise 4.**

Suppose  $f, g: I \rightarrow X$  are paths in a metric space  $X$  such that  $f(1) = g(0)$ . Show that

$$h(x) := \begin{cases} f(2x), & x \in [0, 1/2]; \\ g(2x - 1), & x \in [1/2, 1], \end{cases}$$

is a path in  $X$  from  $f(0)$  to  $g(1)$ .

*Solution.* The map  $h$  is well defined,  $h(0) = f(0)$  and  $h(1) = g(1)$ . We hence need to show that it is continuous.

First, observe that if  $f: [a, b] \rightarrow X$  is continuous, so are  $f_r: [a + r, b + r] \rightarrow X$  and  $f_\lambda: [\lambda a, \lambda b] \rightarrow X$  for any  $r \in \mathbf{R}$  and  $0 \neq \lambda \in \mathbf{R}$ , where  $f_r(x) = f(x - r)$  and  $f_\lambda(x) = f(\frac{x}{\lambda})$ .

Using the above observation, we want to show that if  $f: [0, 1] \rightarrow \mathbf{R}$  and  $g: [1, 2] \rightarrow \mathbf{R}$  are continuous, then so is the map  $h: [0, 2] \rightarrow \mathbf{R}$  defined by  $h(x) = f(x)$  if  $x \leq 1$  and  $h(x) = g(x)$  if  $x \geq 1$ . Let  $U$  be an open subset of  $X$ . Since  $h[0, 2] = f[0, 1] \cup g[1, 2]$  we have  $h^{-1}(U) = f^{-1}(U) \cup g^{-1}(U)$ . The subset  $f^{-1}(U)$  is open in  $[0, 1]$  by continuity of  $f$ . It is thus a union of balls, that is of the form  $\bigcup_{i \in I} (B_i \cap [0, 1])$ , where the  $B_i$  are open intervals of  $\mathbf{R}$ . This implies that  $f^{-1}(U)$  is open in  $[0, 2]$  if and only if it does not contain 1. A similar statement is true for  $g^{-1}(U) = \bigcup_{j \in J} (C_j \cap [1, 2])$  for some open intervals  $C_j$ . If none of  $f^{-1}(U)$  and  $g^{-1}(U)$  contains 1, they are both open in  $[0, 2]$  and so is their union. If one of  $f^{-1}(U)$  or  $g^{-1}(U)$  contains 1 they both do and therefore therefore, for every  $B_i$  containing 1 there exists  $C_j$  (also containing 1) such that  $(B_i \cup C_j) \cap [0, 2]$  is open; and vice-versa for  $C_j$  and  $B_i$ . This implies that  $f^{-1}(U) \cup g^{-1}(U)$  is open in  $[0, 2]$ . ■

**\*Exercise 5.**

- a. Let  $X$  be a metric space and show that: “ $x \sim y$  if there is a path in  $X$  from  $x$  to  $y$ ” is an equivalence relation.
- b. Let  $\pi(X)$  denote the set of equivalence classes (also known as connected components). For any continuous function  $f: X \rightarrow Y$  we define a function

$$\pi(f): \pi(X) \rightarrow \pi(Y)$$

by: if  $C(x) \in \pi(X)$  is the connected component containing  $x \in X$ , then

$$\pi(f)(C(x)) := C(f(x)).$$

Show that  $\pi(f)$  is a well-defined function.

- c. Given continuous functions  $f: X \rightarrow Y$  and  $g: Y \rightarrow Z$  show that the following diagram commutes:

$$\begin{array}{ccc}
 \pi(X) & \xrightarrow{\pi(f)} & \pi(Y) \\
 & \searrow \pi(g \circ f) & \swarrow \pi(g) \\
 & & \pi(Z)
 \end{array}$$

- d. Show that if  $\text{Id}: X \rightarrow X$  is the identity map of  $X$ , then  $\pi(\text{Id}): \pi(X) \rightarrow \pi(X)$  is the identity map of  $\pi(X)$ .
- e. Show that if  $f: X \rightarrow Y$  is a homeomorphism, then  $\pi(f): \pi(X) \rightarrow \pi(Y)$  is a bijection.

*Solution.* a. The constant path  $f: I \rightarrow X$ ,  $f(t) = x$  for all  $t \in I$ , shows that  $x \sim x$ . If  $f: I \rightarrow X$  is a path from  $x$  to  $y$ , then  $g: I \rightarrow X$ ,  $g(t) = f(1-t)$ , gives a path from  $y$  to  $x$ ; so  $x \sim y \iff y \sim x$ . Finally, Question 4 shows that  $x \sim y \& y \sim z \implies x \sim z$ .

b. Suppose that  $y \in C(x)$ , i.e.  $x \sim y$ . We need to show that  $C(f(x)) = C(f(y))$ . Since  $x \sim y$ , we know that  $C := C(x) = C(y)$ , where  $C$  is a connected component of  $X$ . So  $f(x), f(y) \in f(C)$  and  $f(C)$  is the continuous image of a path-connected space and is therefore path-connected. Hence  $C(f(x)) = C(f(y))$ .

c. For any  $x \in X$ ,

$$\begin{aligned}
 (\pi(g) \circ \pi(f))(C(x)) &= \pi(g)(C(f(x))) = C(g(f(x))) \\
 &= C((g \circ f)(x)) = \pi(g \circ f)(C(x)).
 \end{aligned}$$

So  $\pi(g) \circ \pi(f) = \pi(g \circ f)$ .

d. For any  $x \in X$ ,

$$\pi(\text{Id})(C(x)) = C(\text{Id}(x)) = C(x).$$

So  $\pi(\text{Id})$  is the identity map of  $\pi(X)$ .

e. By the previous two parts,

$$\pi(f) \circ \pi(f^{-1}) = \pi(f \circ f^{-1}) = \pi(\text{Id}) = \text{Id},$$

and similarly,  $\pi(f^{-1}) \circ \pi(f) = \text{Id}$ . So  $\pi(f^{-1})$  is the inverse to  $\pi(f)$ , and so  $\pi(f)$  is a bijection. ■

### Exercise 6.

Prove that a discrete metric space is compact if and only if it is finite.

*Solution.* Suppose  $X$  is discrete and compact. Then  $\mathcal{U} := \{\{x\} \mid x \in X\}$  is an open cover of  $X$  and therefore has a finite subcover which implies that  $X$  is finite.

If a discrete space is finite then it is compact as  $\{\{x\} \mid x \in X\}$  is a finite subcover of any open cover of  $X$ . ■

**Exercise 7.**

Given two compact subspaces  $A$  and  $B$  of a metric space  $X$ , prove that  $A \cup B$  and  $A \cap B$  are compact.

Give an example showing that infinite union of compact subsets are not necessarily compact.

*Solution.* Let  $\mathcal{U}$  be an open cover of  $A \cup B$ . Then  $\mathcal{U}$  is an open cover of both  $A$  and  $B$  so there exist finite subcovers  $\mathcal{U}_A$  and  $\mathcal{U}_B$  of  $A$  and  $B$  respectively. So  $\mathcal{U}_A \cup \mathcal{U}_B$  is a finite subcover of  $\mathcal{U}$ .

Now  $A \cap B$  is closed subspace of  $A \cup B$  since  $A$  and  $B$  are both closed being compact. Therefore  $A \cap B$  is a closed subspace of a compact space  $A \cup B$ , and therefore compact.

The intervals  $[-n, n]$  are compact subsets of  $\mathbf{R}$ , but their union is  $\mathbf{R}$  itself, which is not compact. ■

**Exercise 8.**

Let  $(X, d)$  be a compact metric space and suppose that  $f: X \rightarrow X$  is a continuous map such that  $f(x) \neq x$ , for all  $x \in X$ . Prove that there exists  $\varepsilon > 0$  such that  $d(f(x), x) \geq \varepsilon$ , for all  $x \in X$ .

*Hint:* Consider  $g: X \rightarrow \mathbf{R}$ ,  $g(x) = d(f(x), x)$ .

*Solution.* The function  $g: X \rightarrow \mathbf{R}$  given by  $g(x) = d(f(x), x)$  is continuous being a composition of continuous functions ( $g = d \circ (f \times \text{Id}) \circ \Delta$  where  $\Delta(x) = (x, x)$ ). Since  $f(x) \neq x$ , for all  $x \in X$ ,  $g(X) \in (0, +\infty)$ . Since  $X$  is compact,  $g$  attains its bounds on  $X$  and so in particular there exists  $\varepsilon > 0$  such that  $\varepsilon = \inf g(X)$  which gives the result. ■

Exercises marked with a \* are slightly harder bonus questions.

**Exercise 1.**

Prove that the following are equivalent:

- I.  $X$  is compact;
- II. if  $\{V_\alpha \mid \alpha \in I\}$  is a collection of closed subsets of  $X$  such that  $\bigcap_{\alpha \in J} V_\alpha \neq \emptyset$  for any finite  $J \subseteq I$ , then  $\bigcap_{\alpha \in I} V_\alpha \neq \emptyset$ .

*Solution.* “ $\Rightarrow$ ” Suppose that  $X$  is compact and let  $U_\alpha := X \setminus V_\alpha$ . If  $\bigcap_{\alpha \in I} V_\alpha = \emptyset$ , then  $X = X \setminus (\bigcap_{\alpha \in I} V_\alpha) = \bigcup_{\alpha \in I} U_\alpha$ , i.e.  $\{U_\alpha \mid \alpha \in I\}$  is an open cover of  $X$ . Since  $X$  is compact there exists a finite subcover  $\{U_\alpha \mid \alpha \in J\}$ , for some finite subset  $J \subseteq I$ , i.e.

$$X = \bigcup_{\alpha \in J} U_\alpha \quad \Rightarrow \quad X \setminus \left( \bigcup_{\alpha \in J} U_\alpha \right) = \bigcap_{\alpha \in J} V_\alpha = \emptyset,$$

which contradicts the assumption.

“ $\Leftarrow$ ” Let  $\mathcal{U} = \{U_\alpha \mid \alpha \in I\}$  be an open cover of  $X$  and  $V_\alpha := X \setminus U_\alpha$ . Then

$$X = \bigcup_{\alpha \in I} U_\alpha \quad \Rightarrow \quad X \setminus \left( \bigcup_{\alpha \in I} U_\alpha \right) = \bigcap_{\alpha \in I} V_\alpha = \emptyset.$$

So there exists a finite subset  $J \subseteq I$  such that  $\bigcap_{\alpha \in J} V_\alpha = \emptyset$ , which implies that  $X = \bigcup_{\alpha \in J} U_\alpha$ . So  $\{U_\alpha \mid \alpha \in J\}$  is a finite subcover of  $X$ . ■

**Exercise 2.**

Suppose that  $X$  is a compact metric space and  $\{V_n \mid n \in \mathbf{N}\}$  is a collection of non-empty closed nested subsets of  $X$ , that is,  $V_{n+1} \subseteq V_n$  for all  $n \in \mathbf{N}$ . Then  $\bigcap_{n \in \mathbf{N}} V_n \neq \emptyset$ . Find a counterexample with  $X$  non-compact.

*Solution.* Follows from the previous exercise.

Let  $X = (0, 1]$  with the metric induced from the standard metric on  $\mathbf{R}$ . Then the subsets  $(0, 1/n]$  are non-empty closed subsets of  $X$  which are nested, but their intersection is empty. ■

**\*Exercise 3.**

With the notation of Question 2, suppose that  $U$  is open in  $X$  and that  $\bigcap_{n \in \mathbf{N}} V_n \subseteq U$ . Prove that  $V_n \subseteq U$ , for some  $n \in \mathbf{N}$ .

*Hint:* Consider the sequence of sets  $W_n = V_n \cap (X \setminus U)$ .

*Solution.* The  $W_n$  are closed in  $X$  and are also nested. Therefore,  $\bigcap_{n \in \mathbf{N}} W_n \neq \emptyset$  by Question 2, if the  $W_n$  are all non-empty. But  $\bigcap_{n \in \mathbf{N}} W_n = (\bigcap_{n \in \mathbf{N}} V_n) \cap (X \setminus U)$  which implies that  $\bigcap_{n \in \mathbf{N}} V_n \not\subseteq U$  contradicting our assumption. Therefore, at least one of the  $W_n$  must be empty implying that  $V_n \subseteq U$ . ■

**\*Exercise 4.**

Suppose that  $X$  is a compact metric space and suppose that  $f: X \rightarrow X$  is a continuous map. Let  $X_0 = X$ ,  $X_1 = f(X_0)$  and inductively define  $X_{n+1} = f(X_n)$  for  $n \geq 1$ .

- a. Show that  $A = \bigcap_{n \in \mathbf{N}} X_n$  is non-empty.

*Hint: Remember Question 2.*

- b. Show further that  $f(A) = A$ .

*Hint: To show that  $a \in A$  is in  $f(A)$  apply Question 2 to the sets  $V_n = f^{-1}(a) \cap X_n$ .*

*Solution.* a. We show that the  $X_n$  form a nested sequence of closed subsets of  $X$ . Note that  $X_1 = f(X_0) = f(X) \subseteq X_0$ . Suppose inductively that  $X_n \subseteq X_{n-1}$  for some integer  $n \geq 1$ . Then  $f(X_n) \subseteq f(X_{n-1})$  which says  $X_{n+1} \subseteq X_n$ . So by induction  $X_n \subseteq X_{n-1}$  for all  $n \in \mathbf{N}$ . Also, since  $X$  is compact and  $f$  is continuous,  $X_1 = f(X)$  is compact. Suppose that  $X_n$  is compact for some integer  $n \geq 0$ . Then since  $f$  is continuous,  $X_{n+1} = f(X_n)$  is also compact. By induction  $X_n$  is compact for all integers  $n \geq 0$ . So each  $X_n$  is closed in  $X$ . Also, each  $X_n$  is non-empty by inductive construction. Now by Question 2,  $A = \bigcap_{n \in \mathbf{N}} X_n$  is non-empty.

- b. The inclusion  $f(A) \subseteq A$  is straightforward: if  $a \in A$  then  $a \in X_n$  for any integer  $n \geq 0$ , so  $f(a) \in f(X_n) = X_{n+1}$  for any integer  $n \geq 0$ . But since  $X_0 \supseteq X_1 \supseteq \dots \supseteq X_n \supseteq \dots$ , we have  $\bigcap_{n=0}^{\infty} X_{n+1} = \bigcap_{n=1}^{\infty} X_n = \bigcap_{n=0}^{\infty} X_n = A$ , and we see that  $f(a) \in A$ . To prove the other inclusion we follow the hint: for any  $a \in A$  let  $V_n = f^{-1}(a) \cap X_n$ . Since the  $X_n$  are nested so are the  $V_n$ . Then  $\{a\}$  is closed in  $A$ , hence since  $f$  is continuous,  $f^{-1}(a)$  is closed in  $X$ . Also each  $X_n$  is closed in  $X$  as in a. So each  $V_n$  is closed in  $X$ . Moreover, for each integer  $n \geq 0$  we know  $a \in X_{n+1} = f(X_n)$  so there exists  $x \in X_n$  such that  $f(x) = a$ . This says that  $V_n = f^{-1}(a) \cap X_n \neq \emptyset$ . Now by Question 2,  $\bigcap_{n=0}^{\infty} V_n$  is non-empty. Let  $b$  be a point in this set. Then  $b \in V_n = f^{-1}(a) \cap X_n$  for any integer  $n \geq 0$ . Also,  $b = f^{-1}(a)$  says that  $f(b) = a$ , and  $b \in X_n$  for all integers  $n \geq 0$  says that  $b \in \bigcap_{n=0}^{\infty} X_n = A$ . So  $A \subseteq f(A)$ . We have now proved that  $f(A) = A$ . ■

**Exercise 5.**

Which of the following subspaces of  $\mathbf{R}$  and  $\mathbf{R}^2$  are compact?

- a.  $[0, 1)$ ;

- b.  $[0, \infty)$ ;
- c.  $\mathbf{Q} \cap [0, 1]$ ;
- d.  $\{(x, y) \in \mathbf{R}^2 \mid x^2 + y^2 = 1\}$ ;
- e.  $\{(x, y) \in \mathbf{R}^2 \mid |x| + |y| \leq 1\}$ ;
- f.  $\{(x, y) \in \mathbf{R}^2 \mid x^2 + y^2 < 1\}$ ;
- g.  $\{(x, y) \in \mathbf{R}^2 \mid x \geq 1, 0 \leq y \leq 1/x\}$ .

*Solution.* d and e are compact. The others are not. ■

#### Exercise 6.

Finish the proof of Corollary 3.2.5. More precisely, shows that if  $f: S \rightarrow \mathbf{R}$  is a bounded function ( $S$  any set, not necessarily a metric space), then  $\sup f(X)$  and  $\inf f(X)$  are elements of  $\overline{f(X)}$ .

*Solution.* By Analysis 1, we have a converging sequence  $f(x_n) \rightarrow \sup f(X)$ , which implies  $\sup f(X) \in \overline{\{f(x_n) \mid n \in \mathbf{N}\}} \subseteq \overline{f(X)}$ . The proof for the infimum is similar. ■

#### Exercise 7.

In the proof of the Heine-Borel Theorem we used the following fact: if  $A \subseteq \mathbf{R}^n$  is bounded, then there exists  $a, b \in \mathbf{R}$  such that  $A \subseteq [a, b]^n$ . Prove this.

*Solution.* If  $A$  is bounded, then there exists  $r > 0$  such that  $A \subseteq \overline{B}_r(x)$ , for some  $x \in A$ . Taking  $a = \min\{x_i - r \mid i = 1, 2, \dots, n\}$  and  $b = \max\{x_i + r \mid i = 1, 2, \dots, n\}$  we see that  $\overline{B}_r(x) \subseteq [a, b]^n$ . ■

Exercises marked with a \* are slightly harder bonus questions.

**Exercise 1.**

Suppose that  $f: (0, 1) \rightarrow \mathbf{R}$  is continuous, monotonic and bounded on  $(0, 1)$ . Prove that  $f$  is uniformly continuous.

*Solution.* Suppose that  $f$  is increasing. Then we can prove

$$\lim_{x \rightarrow 0^+} f(x) = \inf_{x \in (0,1)} f(x) \quad \text{and} \quad \lim_{x \rightarrow 1^-} f(x) = \sup_{x \in (0,1)} f(x).$$

We can now define  $\bar{f}: [0, 1] \rightarrow \mathbf{R}$  by

$$\bar{f}(x) := \begin{cases} \inf f(x), & x = 0; \\ f(x), & x \in (0, 1); \\ \sup f(x), & x = 1. \end{cases}$$

Then  $\bar{f}$  is a continuous function from a compact domain and so is uniformly continuous. Therefore,  $f = \bar{f}|_{(0,1)}$  is uniformly continuous. ■

**Exercise 2.**

Show that two Lipschitz equivalent metrics are uniformly equivalent.

*Solution.* Suppose that  $d_1$  and  $d_2$  are Lipschitz equivalent metrics on  $X$ , i.e. there exists  $h, k \in (0, \infty)$  such that

$$hd_2(x, y) \leq d_1(x, y) \leq kd_2(x, y), \quad \forall x, y \in X.$$

Then we need to show that for every  $\varepsilon > 0$ , there exists  $\delta_1, \delta_2 > 0$  such that

$$d_1(x, y) < \delta_1 \implies d_2(x, y) < \varepsilon \quad \text{and} \quad d_2(x, y) < \delta_2 \implies d_1(x, y) < \varepsilon.$$

We can take  $\delta_1 = h\varepsilon$  and  $\delta_2 = \varepsilon/k$ . ■

**Exercise 3.**

Show that for any metric space  $(X, d)$ , the metrics  $d$  and  $d'(x, y) := \frac{d(x, y)}{1+d(x, y)}$  are uniformly equivalent but may not be Lipschitz equivalent.

*Solution.* Note that  $0 \leq d'(x, y) < 1$ , for all  $x, y \in X$ . We need to show that for all  $\varepsilon > 0$ , there exists  $\delta, \delta' > 0$  such that

$$d(x, y) < \delta \implies d'(x, y) < \varepsilon \quad \text{and} \quad d'(x, y) < \delta' \implies d(x, y) < \varepsilon.$$

We can take  $\delta = \varepsilon$  and  $\delta' = \varepsilon/(\varepsilon + 1)$  since

$$d'(x, y) \leq d(x, y) < \delta \quad \text{and} \quad d(x, y) = \frac{d'(x, y)}{1 - d'(x, y)} < \frac{\delta'}{1 - \delta'}.$$

On  $\mathbf{R}$ , if  $d$  and  $d'$  were Lipschitz equivalent then there would exist  $k > 0$  such that

$$d(x, y) = |x - y| \leq kd'(x, y) < k, \quad \forall x, y \in \mathbf{R},$$

which is obviously false. ■

#### Exercise 4.

Show that the metric on  $(0, 1)$  given by  $d(x, y) := |x^{-1} - y^{-1}|$  is topologically equivalent but not uniformly equivalent to the Euclidean metric on  $(0, 1)$ .

*Solution.* Let  $d_2$  denote the Euclidean metric on  $(0, 1)$ . Since  $d_2(x, y) = |x - y| \leq d(x, y)$ , if a subset  $U \subseteq (0, 1)$  is  $d_2$ -open, then it is  $d$ -open. Given  $a \in (0, 1)$  and  $\varepsilon > 0$ , let  $\delta = \min\{\frac{1}{2}a^2\varepsilon, a/2\}$ . If  $|x - a| < \delta$ , then

$$d(x, a) = \frac{|x - a|}{|xa|} < \frac{2|x - a|}{a^2} < \varepsilon.$$

Therefore, the identity map is  $(d_2, d)$ -continuous, i.e. if  $U$  is  $d$ -open, then it is  $d_2$ -open. So  $d$  and  $d_2$  are topologically equivalent.

But the identity map is not uniformly  $(d_2, d)$ -continuous, since taking  $\varepsilon = 1$  it is not possible to find a  $\delta > 0$  such that  $|x - y| < \delta$  implies  $d(x, y) < 1$ . ■

#### Exercise 5.

Prove that if  $A \subseteq \mathbf{R}$  is not closed, then there exists a continuous function  $f: A \rightarrow \mathbf{R}$  that is not uniformly continuous.

*Solution.* If  $A$  is not closed then there exists a point  $x \in \bar{A} \setminus A$ . Define  $f: A \rightarrow \mathbf{R}$  by  $f(x) = 1/x - a$ . Then this is a continuous function that is not uniformly continuous. ■

#### Exercise 6.

Prove that any Cauchy sequence in a metric space is bounded.

*Solution.* Let  $(x_n)$  be a Cauchy sequence in a metric space  $(X, d)$  and take  $\varepsilon = 1$ . Then there exists  $N \in \mathbf{N}$  such that  $m, n \geq N \implies d(x_m, x_n) < 1$ . Let

$$M := \max\{d(x_i, x_N) \mid i \in \{1, \dots, N - 1\}\}.$$

Then  $d(x_n, x_N) \leq \max\{1, M\}$ , for all  $n \in \mathbf{N}$ , and so  $(x_n)$  is bounded. ■

**Exercise 7.**

Give an example of a metric space that is totally bounded but not compact.

*Solution.* The subspace  $(0, 1) \subseteq \mathbf{R}$  is totally bounded (in general, bounded and totally bounded are the same in Euclidean space) but it is not compact. ■

**Exercise 8.**

Find appropriate sequences that prove the following subspaces of Euclidean space are not sequentially compact:

- $(-6, 9) \subseteq \mathbf{R}$ ;
- $\mathbf{Z}^2 \subseteq \mathbf{R}^2$ ;
- $\{x \in \mathbf{R}^3 \mid |x_1| > 2\} \subseteq \mathbf{R}^3$ ;
- $\{x \in \mathbf{R}^4 \mid x_1 x_2 x_3 x_4 < 0\} \subseteq \mathbf{R}^4$ .

*Solution.* In each subspace we need to find an infinite sequence with no convergent subsequences (in that subspace!) The following are suitable examples (amongst others):

- $(1/n - 6)$ ;
- $(n, n)$ ;
- $(3, 0, n)$ ;
- $(n, -1, 1, 10)$ .

**\*Exercise 9.**

Suppose that  $X$  is a sequentially compact metric space and that  $\mathcal{U} = \{U_1, U_2, \dots, U_n\}$  is an open cover of  $X$ . Put  $C_i := X \setminus U_i$ . Show that if  $U_i = X$ , for some  $i \in \{1, \dots, n\}$ , then any  $\varepsilon > 0$  is a Lebesgue number for  $\mathcal{U}$ . From now on, assume that no  $U_i$  is  $X$ .

- Defining  $d(x, C_i) = \inf\{d(x, c) \mid c \in C_i\}$ , prove that  $x \mapsto d(x, C_i)$  is a continuous real-valued function on  $X$  all of whose values are non-negative.
- Defining  $f(x) = \frac{1}{n} \sum_{i=1}^n d(x, C_i)$ , show that  $f: X \rightarrow \mathbf{R}$  is continuous with a positive value for each  $x \in X$ .
- Deduce that there exists  $\varepsilon > 0$  such that  $f(x) \geq \varepsilon$ , for all  $x \in X$ .
- For any  $x \in X$ , show that  $f(x) \leq \max\{d(x, C_i) \mid i \in \{1, \dots, n\}\}$ .
- For any  $x \in X$ , show that  $B_\varepsilon(x) \subseteq U_{k(x)}$ , where  $k(x) \in \{1, \dots, n\}$  is such that  $d(x, C_{k(x)}) = \max\{d(x, C_i) \mid i \in \{1, \dots, n\}\}$ . Deduce that  $\varepsilon$  is a Lebesgue number for  $\mathcal{U}$ .

*Solution.* We can just take  $U(x) = U_i$ , for all  $x \in X$ , to show that any  $\varepsilon > 0$  is a Lebesgue number if  $U_i = X$ .

- a. Since  $C_i$  is closed in  $X$ , it is compact, and for every  $x \in X$ , there is a  $c_x \in C_i$  such that  $d(x, C_i) = d(x, c_x)$ . Therefore,  $d(x, c_x) \leq d(x, c)$ , for every  $c \in C_i$ . Now let  $x \in X$  and  $\varepsilon > 0$ . Take  $\delta = \varepsilon$  and suppose  $d(x, y) < \delta$ . Suppose, without loss of generality, that  $d(x, C_i) \geq d(y, C_i)$ . Then

$$|d(x, C_i) - d(y, C_i)| = d(x, c_x) - d(y, c_y) \leq d(x, c_y) - d(y, c_y) \leq d(x, y) < \delta = \varepsilon.$$

So the function is continuous. Since  $d(x, C_i)$  is the infimum of a set of non-negative numbers it is also non-negative.

- b. By the composition of continuous functions,  $f$  is continuous. For  $x \in X$ ,  $x$  cannot be in every  $C_i$  since otherwise  $x$  would not be in any  $U_i$  and  $\mathcal{U}$  would not be an open cover for  $X$ . Therefore, there exists  $k \in \{1, \dots, n\}$  such that  $x \notin C_k$  and  $d(x, C_k) > 0$ . Therefore,  $f(x) > 0$ , for all  $x \in X$ .
- c. Since  $f$  is a continuous real-valued function with compact domain it attains its bounds, in particular, its infimum. As it is positive everywhere, this implies that there exists  $\varepsilon > 0$  such that  $f(x) \geq \varepsilon$  for all  $x \in X$ .
- d. We have,

$$\begin{aligned} f(x) &= \frac{1}{n} \sum_{i=1}^n d(x, C_i) \leq \frac{1}{n} (n \cdot \max\{d(x, C_i) \mid i \in \{1, \dots, n\}\}) \\ &= \max\{d(x, C_i) \mid i \in \{1, \dots, n\}\}. \end{aligned}$$

- e. Suppose that  $y \in B_\varepsilon(x)$ , that is:  $d(x, y) < \varepsilon$ . We need to show that  $y \in U_{k(x)} = X \setminus C_{k(x)}$ . We know that  $0 < \varepsilon \leq f(x) \leq d(x, C_{k(x)})$ . If  $y \in C_{k(x)}$ , then  $d(x, C_{k(x)}) < d(x, y) < \varepsilon$  providing a contradiction. ■

**\*Exercise 10.**

Suppose that  $X$  is a compact metric space and  $V_1 \supseteq V_2 \supseteq \dots$  is a nested sequence of closed subsets of  $X$ . Prove that

$$\text{diam} \left( \bigcap_{n=1}^{\infty} V_n \right) = \inf\{\text{diam } V_n \mid n \in \mathbf{N}\}.$$

*Solution.* If, say,  $V_{n_0}$  is empty, then  $\bigcap_{n=1}^{\infty} V_n = \emptyset$  and its diameter is 0 by definition. Likewise in this case  $\text{diam } V_{n_0} = 0$  so  $\inf\{\text{diam } V_n \mid n \in \mathbf{N}\} = 0$  also. Suppose now that all the  $V_n$  are non-empty. (We already know that their intersection is then non-empty.) Now  $\bigcap_{n=1}^{\infty} V_n \subseteq V_m$  for any  $m \in \mathbf{N}$  so  $\text{diam} \left( \bigcap_{n=1}^{\infty} V_n \right) \leq \text{diam } V_m$ . Hence  $\text{diam} \left( \bigcap_{n=1}^{\infty} V_n \right) \leq \inf\{\text{diam } V_m \mid m \in \mathbf{N}\} = m_0$  say.

Conversely  $m_0$  is a lower bound for the diameters of the  $V_n$  so for any  $\varepsilon > 0$  and any  $n \in \mathbf{N}$  we know that  $\text{diam } V_n > m_0 - \varepsilon$ . Hence there exist points  $x_n, y_n \in V_n$  such

that  $d(x_m, x_n) > m_0 - \varepsilon$ . Since  $X$  is sequentially compact  $(x_n)$  has a subsequence  $(x_{n_r})$  converging to a point  $x \in X$ , and then  $(y_{n_r})$  has a subsequence  $(y_{n_{r_s}})$  converging to a point  $y \in X$ . Since  $(x_{n_{r_s}})$  is a subsequence of  $(x_{n_r})$  it too converges to  $x$ . Also, by continuity of the metric,  $d(x_{n_{r_s}}, y_{n_{r_s}}) \rightarrow d(x, y)$  as  $s \rightarrow \infty$ . Hence  $d(x, y) \geq m_0 - \varepsilon$ . Also,  $x, y \in V_n$  for each  $n \in \mathbf{N}$  since  $V_n$  is closed in  $X$ . Since this is true for all  $n \in \mathbf{N}$  we have  $x, y \in \bigcap_{n=1}^{\infty} V_n$ . Hence  $\text{diam} \bigcap_{n=1}^{\infty} V_n \geq m_0 - \varepsilon$ . But this is true for any  $\varepsilon > 0$ , so  $\text{diam} \bigcap_{n=1}^{\infty} V_n \geq m_0$ . The above taken together proves the result. ■

Exercises marked with a \* are slightly harder bonus questions.

**Exercise 1.**

Show that in any metric space

- The union of two complete subspaces is complete.
- The intersection of two complete subspaces is complete.

*Hint: Use Lemma 4.1.5 for part a.*

*Solution.* a. Let  $X$  and  $Y$  be complete metric spaces and  $(a_n)$  be a Cauchy sequence in  $X \cup Y$ . Then  $(a_n)$  has infinitely many terms in either  $X$  or  $Y$  (as an infinite sequence). Suppose  $(a_n)$  has infinitely many terms in  $X$ , then these terms can be considered as a subsequence of  $(a_n)$  in  $X$  which will converge to a point in  $X$  by completeness. Therefore,  $(a_n)$  converges in  $X \cup Y$  by Lemma 4.1.5.

- The intersection  $X \cap Y$  is a closed subspace of the complete space  $X$ , and is therefore complete. ■

**Exercise 2.**

Prove that any discrete metric space is complete.

*Solution.* Let  $(x_n)$  be a Cauchy sequence in a discrete metric space  $X$ . Taking  $\varepsilon = 1$  in the definition of a Cauchy sequence proves that  $(x_n)$  is eventually constant and therefore converges. ■

**Exercise 3.**

Which of the following are complete:

- $\{\frac{1}{n} \mid n \in \mathbf{N}\} \cup \{0\}$ ;
- $\mathbf{Q} \cap [0, 1]$ ;
- $\{(x, y) \in \mathbf{R}^2 \mid x > 0, y \geq \frac{1}{x}\}$ .

*Solution.* a. Complete (as a closed subspace of  $\mathbf{R}$ );

- Not complete;

c. Complete (as a closed subspace of  $\mathbf{R}^2$ ). ■

**Exercise 4.**

Give an example of topologically equivalent metrics  $d$  and  $d'$  on a set  $X$  and a sequence  $(x_n)$  in  $X$  that is Cauchy in  $(X, d)$  but not in  $(X, d')$ .

*Hint: On a previous exercise sheet we had an example of topologically equivalent metrics that were not uniformly equivalent.*

*Solution.* On  $(0, 1)$ , take the standard metric  $d_2$  and  $d(x, y) = |x^{-1} - y^{-1}|$ . Then the sequence  $(1/n)$  is  $d_2$ -Cauchy but is not  $d$ -Cauchy. ■

**\*Exercise 5.**

Which of the following metrics on  $\mathbf{R}$  give complete metric spaces?

- $d(x, y) = |x^3 - y^3|$ ;
- $d(x, y) = |e^x - e^y|$ ;
- $d(x, y) = |\tan^{-1}(x) - \tan^{-1}(y)|$ .

*Solution.* Let us label these metrics  $d_a, d_b, d_c$ .

- Then  $(\mathbf{R}, d_a)$  is complete. For let  $(x_n)$  be a  $d_a$ -Cauchy sequence. Then  $(x_n^3)$  is a Cauchy sequence in the ordinary sense, so it converges say to  $y \in \mathbf{R}$ . Let  $x = y^{\frac{1}{3}}$ . Then  $(x_n)$  converges to  $x$  in the metric  $d_a$ . This holds since

$$d_a(x_n, x) = |x_n^3 - x^3| = |x_n^3 - y| \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

- This is not complete. Consider the sequence  $(-n)$ . Then  $e^{-n} \rightarrow 0$  as  $n \rightarrow \infty$ , so  $(e^{-n})$  is Cauchy in the usual sense, so  $(-n)$  is  $d_b$ -Cauchy. But suppose that  $(-n)$  converged to a real number  $x_0$  in  $(\mathbf{R}, d_b)$ . Then  $d_b(-n, x_0) \rightarrow 0$  as  $n \rightarrow \infty$ , which says that  $(e^{-n})$  converges to  $e^{x_0}$  in the usual metric on  $\mathbf{R}$ . But we know that  $(e^{-n})$  converges to 0 in the usual metric, so would have  $e^{x_0} = 0$ . But there is no such real number  $x_0$ , so  $(-n)$  cannot converge in  $(\mathbf{R}, d_b)$ . Hence  $(\mathbf{R}, d_b)$  is not complete.
- For similar reasons this is not complete either. Consider the sequence  $(n)$  in  $\mathbf{R}$ . The sequence  $(\tan^{-1}(n))$  converges to  $\pi/2$  in the usual sense as  $n \rightarrow \infty$ . So  $(\tan^{-1}(n))$  is Cauchy in the usual sense which says that  $(n)$  is  $d_c$ -Cauchy. But suppose that  $(n)$  converges to some  $x_0 \in \mathbf{R}$  in the metric space  $(\mathbf{R}, d_c)$ . Then  $(\tan^{-1}(n))$  converges to  $\tan^{-1}(x_0)$  in the usual sense. This gives  $\tan^{-1}(x_0) = \pi/2$ . But there is no such real number  $x_0$ . So  $(n)$  does not converge in  $(\mathbf{R}, d_c)$  and this space is not complete. ■

**Exercise 6.**

Show that the uniformly continuous image of a complete metric space is not necessarily complete.

*Solution.* Define  $f: [1, +\infty) \rightarrow (0, 1]$  by  $f(x) = 1/x$ . ■

**\*Exercise 7.**

Suppose that  $f: X \rightarrow Y$  is a bijection between metric spaces such that  $f$  is uniformly continuous and  $f^{-1}$  is continuous.

- a. Prove that if  $(x_n)$  is a Cauchy sequence in  $X$ , then  $f(x_n)$  is a Cauchy sequence in  $Y$ .
- b. Prove that if  $y_n$  is a convergent sequence in  $Y$  then  $(f^{-1}(y_n))$  is a convergent sequence in  $X$ .
- c. Deduce that if  $Y$  is complete, then  $X$  is complete.
- d. Deduce that completeness is an invariant of uniform equivalence.

*Solution.* a. Let  $\varepsilon > 0$ . Then, by the uniform continuity of  $f$ , there is a  $\delta > 0$  such that  $d_X(x, y) < \delta \implies d_Y(f(x), f(y)) < \varepsilon$ . For this  $\delta$ , there is an  $N \in \mathbf{N}$ , such that  $m, n \geq N \implies d_X(x_m, x_n) < \delta$  since  $(x_n)$  is Cauchy. Therefore,  $m, n \geq N \implies d_Y(f(x_m), f(x_n)) < \varepsilon$ .

- b. Follows from the (sequential) continuity of  $f^{-1}$ .
- c. Let  $(x_n)$  be a Cauchy sequence in  $X$ . Then  $f(x_n)$  is a Cauchy sequence in  $Y$  by a which implies that  $f(x_n)$  converges in  $Y$  since we are assuming  $Y$  is complete. Therefore, by b,  $(f^{-1}(f(x_n))) = (x_n)$  converges in  $X$  and so  $X$  is complete.
- d. If we suppose that  $f$  is now a uniform equivalence we immediately obtain that if  $Y$  is complete, then  $X$  is complete. By considering  $f^{-1}$  we also get the converse. ■

**Exercise 1.**

Let  $f: (0, 1/4) \rightarrow (0, 1/4)$  be given by  $f(x) = x^2$ . Show that  $f$  is a contraction without a fixed point.

*Solution.* Firstly,  $f(0, 1/4) \subseteq (0, 1/4)$ . Also,  $f$  is a contraction since

$$|f(x) - f(y)| = |x^2 - y^2| = (x + y) \cdot |x - y| < \frac{1}{2}|x - y|.$$

It has no fixed point since  $x^2 = x \implies x = 1 \notin (0, 1/4)$ . ■

**Exercise 2.**

Let  $X = \{x, y, z\}$  and let  $d$  be the discrete metric on  $X$ . Define

$$d'(x, y) = 2 = d'(x, z), \quad d'(y, z) = 1, \quad \text{and} \quad d'(x, x) = d'(y, y) = d'(z, z) = 0.$$

Show that  $d'$  is a metric on  $X$  that is Lipschitz equivalent to  $d$ . Define  $f: X \rightarrow X$  by

$$f(x) = y, \quad f(y) = z \quad \text{and} \quad f(z) = z.$$

Show that  $f$  is a  $d'$ -contraction but not a  $d$ -contraction.

*Solution.* It is easy to show that  $d'$  is a metric on  $X$ . To show that it is Lipschitz equivalent to  $d$  note that

$$\frac{1}{2}d'(a, b) \leq d(a, b) \leq d'(a, b) \quad \forall a, b \in X.$$

By putting  $K = 1/2$  we can see that  $f$  is a  $d'$ -contraction by checking the definition. To show that  $f$  is not a  $d$ -contraction observe that

$$d(f(x), f(y)) = d(y, z) = 1 \leq Kd(x, y) = K \implies K \geq 1.$$

Since  $K$  has to be less than 1,  $f$  is not a  $d$ -contraction. ■

**Exercise 3.**

Let  $f: X \rightarrow X$  be a map of a compact metric space  $X$  such that

$$d(f(x), f(y)) < d(x, y)$$

for any distinct points  $x, y \in X$ . Prove that  $f$  has a unique fixed point.

*Hint:* Show that  $\inf\{d(x, f(x)) \mid x \in X\}$  is attained, and get a contradiction unless this inf is zero.

add ref for the hint

*Solution.* Consider the composition

$$d \circ (1 \times f) \circ \Delta: X \rightarrow X \times X \rightarrow X \times X \rightarrow \mathbf{R}.$$

Each map here is continuous so the composition,  $F$  say, is continuous. Since  $X$  is compact,  $F$  attains its lower bound  $l$ , say, on  $X$ . Suppose that  $l > 0$  and that it is attained at  $x_0$ , so  $d(x_0, f(x_0)) = l$ . Then  $x_0$  and  $f(x_0)$  are distinct, so by assumption  $l \leq F(f(x_0)) = d(f(x_0), f(f(x_0))) < d(x_0, f(x_0)) = l$ . This contradiction shows that  $l$  must be 0. Since  $l$  is attained  $d(p, f(p)) = 0$  for some  $p \in X$ , so  $f(p) = p$  and we have proved the existence of a fixed point.

Uniqueness: for distinct fixed points  $p, q$  we would have

$$d(p, q) = d(f(p), f(q)) < d(p, q),$$

a contradiction. So  $p$  is the unique fixed point. ■

#### Exercise 4.

Give examples of:

- a nested sequence of non-empty closed sets  $V_n \subseteq (0, 1)$  with  $\text{diam } V_n \rightarrow 0$ , as  $n \rightarrow \infty$ , but  $\bigcap_{n \in \mathbf{N}} V_n = \emptyset$ ;
- a nested sequence of non-empty sets  $U_n$  in  $\mathbf{R}$  with  $\text{diam } U_n \rightarrow 0$ , as  $n \rightarrow \infty$ , but  $\bigcap_{n \in \mathbf{N}} U_n = \emptyset$ ;
- a nested sequence of non-empty closed sets  $W_n \subseteq \mathbf{R}$  with  $\bigcap_{n \in \mathbf{N}} W_n = \emptyset$ .

*Solution.* a.  $(V_n) = ((0, 1/n])_{n \geq 2}$ ;

b.  $(U_n) = ((0, 1/n))_{n \geq 2}$ ;

c.  $(W_n) = ([n, +\infty))_{n \geq 1}$ . ■

#### Exercise 5.

Determine whether or not the real numbers  $8/81, 1/4, 0.296296296 \dots, \pi/4$  and  $\sqrt{3}/2$  lie in the Cantor Set  $K \subseteq \mathbf{R}$ .

*Solution.* To compute the ternary expansion of  $x = 8/81$ , write

$$x = (2 \times 3) + 2/81 = 2/27 + 2/81 = 0.002200 \dots ;$$

since there are no 1s, it follows that  $8/81 \in K$ .

For the ternary expansion  $x = 1/4 = 0.x_1x_2x_3 \dots$ , note that  $x_j$  are given by iterating the following procedure: multiply  $x$  by 3, record the integer part, take the fractional part,

and repeat. So  $3/4 = x_1 \cdot x_2 x_3 \dots$ , and so  $x_1 = 0$ ; then  $\times 3$  gives  $2\frac{1}{4} = 9/4 = x_2 \cdot x_3 x_4 \dots$ , so  $x_2 = 2$ , and  $1/4 = 0.x_3 x_4 \dots$ . Repeating forever gives  $1/4 = 0.202020 \dots$ , which therefore lies in  $K$ .

For  $x = 0.296296296 \dots$ , note that  $10^3 x = 296 + x$ , so  $x = 296/999 = 8/27$ . The ternary expansion is given by

$$x = (2 \times 3) + 2/27 = 2/9 + 2/27 = 0.02200 \dots,$$

so  $0.296296296 \dots \in K$ .

For  $x = \pi/4$  and  $\sqrt{3}/2$ , follow the same procedure as for  $1/4$ . So  $x = \pi/4 = 0.785398 \dots$  implies that  $x_1$  is the integer part of  $3x = 2.35619 \dots$ , and  $x_2$  is the integer part of  $3 \times 0.35619 \dots = 1.0685 \dots$ . Since  $\pi/4$  is irrational,  $x_3 x_4 \dots$  cannot give  $22 \dots$ . Thus the ternary expansion begins  $0.21 \dots$ , and  $\pi/4 \notin K$ .

Similarly,  $x = \sqrt{3}/2 = 0.866025 \dots$  implies that  $x_1$  is the integer part of  $3x = 2.59807 \dots$ , and  $x_2$  is the integer part of  $3 \times 0.59807 \dots = 1.7942 \dots$ . Since  $\sqrt{3}/2$  is irrational,  $x_3 x_4 \dots$  cannot give  $22 \dots$ . Thus the ternary expansion also begins  $0.21 \dots$ , and  $\sqrt{3}/2 \notin K$ . ■

### Exercise 6.

The set  $Av \subseteq K$  of averages in the Cantor Set is defined by

$$Av := \left\{ \frac{a+b}{2} \mid a, b \in K, a \neq b \right\} \cap K \subseteq \mathbf{R}.$$

Prove that  $Av = \{k/3^m \mid m \geq 1, 1 \leq k \leq 3^m, k \not\equiv 0 \pmod{3}\} \cap K$ .

*Solution.* Let  $a, b \in K$  have ternary expansions  $0.a_1 a_2 \dots, 0.b_1 b_2 \dots$ , where  $a \neq b$ . Because  $a_n, b_n \neq 1$  for any  $n$ , it follows that  $a_n + b_n/2 = 0, 1$  or  $2$ , and therefore that the ternary expansion of  $c := (a+b)/2$  has  $c_n = (a_n + b_n)/2$ . Moreover, there exists a least value of  $m$  for which  $a_m \neq b_m$ , and so  $c_m = 1$ . So  $c \in K$  implies that  $c_j = 0$  or  $2$  for all  $j > m$ ; if the former, then  $a_j = b_j = 0$ , and if the latter, then  $a_j = b_j = 2$ .

In other words,  $c \in Av$  if and only if it has a finite ternary expansion that ends with  $1$  or  $2$  in the  $m$ th place, and otherwise contains only  $0$ s or  $2$ s. Therefore,  $c = k/3^m$  for some  $k$  not divisible by  $3$ , where  $1 \leq k \leq 3^m$ . ■

### Exercise 7.

Consider the following five points in  $(\mathbf{R}^2, d_2)$ :

$$x_1 = (-3, 1), x_2 = (-3, 2), x_3 = (-1, 1), x_4 = (3, -1), x_5 = (3, 2).$$

Perform the  $k$ -means clustering algorithm setting  $k = 2$  by using  $x_1$  and  $x_2$  as your initial means.

Write solution

**Exercise 8.**

Let  $X$  be the set of points in Question 7. Calculate the relative distance between the two subsets obtained in Question 7. Are these subsets cohesive and do they form clusters in  $X$ ?

Write solution

Exercises marked with a \* are slightly harder bonus questions.

**Exercise 1.**

Show that in any metric space

- The union of two complete spaces is complete. *Hint: Use Lemma 1.5.*
- The intersection (provided it is non-empty) of two complete spaces is complete.

ref

*Solution.* a. Let  $X$  and  $Y$  be complete metric spaces and  $(a_n)$  be a Cauchy sequence in  $X \cup Y$ . Then  $(a_n)$  has infinitely many terms in either  $X$  or  $Y$  (as an infinite sequence). Suppose  $(a_n)$  has infinitely many terms in  $X$ , then these terms can be considered as a subsequence of  $(a_n)$  in  $X$  which will converge to a point in  $X$  by completeness. Therefore,  $(a_n)$  converges in  $X \cup Y$  by Lemma 1.5. ref

- The intersection  $X \cap Y$  is a closed subspace of the complete space  $X$ , and is therefore complete. ■

**Exercise 2.**

Prove that any discrete metric space is complete.

*Solution.* Let  $(x_n)$  be a Cauchy sequence in a discrete metric space  $X$ . Taking  $\varepsilon = 1$  in the definition of a Cauchy sequence proves that  $(x_n)$  is eventually constant and therefore converges. ■

**Exercise 3.**

Which of the following are complete:

- $\{\frac{1}{n} \mid n \in \mathbf{N}\} \cup \{0\}$ ;
- $\mathbf{Q} \cap [0, 1]$ ;
- $\{(x, y) \in \mathbf{R}^2 \mid x > 0, y \geq \frac{1}{x}\}$ .

*Solution.* a. Complete (as a closed subspace of  $\mathbf{R}$ );

- Not complete (take any a Cauchy sequence of rationals converging to  $\sqrt{2}-1 \in [0, 1]$ );

c. Complete (as a closed subspace of  $\mathbf{R}^2$ ). ■

**Exercise 4.**

Give an example of topologically equivalent metrics  $d$  and  $d'$  on a set  $X$  and a sequence  $(x_n)$  in  $X$  that is Cauchy in  $(X, d)$  but not in  $(X, d')$ .

*Hint: On a previous exercise sheet we had an example of topologically equivalent metrics that were not uniformly equivalent.*

*Solution.* On  $(0, 1)$ , take the standard metric  $d_2$  and  $d(x, y) = |x^{-1} - y^{-1}|$ . Then the sequence  $(\frac{1}{n})$  is  $d_2$ -Cauchy but is not  $d$ -Cauchy. ■

**\*Exercise 5.**

Which of the following metrics on  $\mathbf{R}$  give complete metric spaces?

- $d(x, y) = |x^3 - y^3|$ ;
- $d(x, y) = |e^x - e^y|$ ;
- $d(x, y) = |\tan^{-1}(x) - \tan^{-1}(y)|$ .

*Solution.* Let us label these metrics  $d_a, d_b, d_c$ .

- Then  $(\mathbf{R}, d_a)$  is complete. For let  $(x_n)$  be a  $d_a$ -Cauchy sequence. Then  $(x_n^3)$  is a Cauchy sequence in the ordinary sense, so it converges say to  $y \in \mathbf{R}$ . Let  $x = y^{\frac{1}{3}}$ . Then  $(x_n)$  converges to  $x$  in the metric  $d_a$ . This holds since

$$d_a(x_n, x) = |x_n^3 - x^3| = |x_n^3 - y| \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

- This is not complete. Consider the sequence  $(-n)$ . The sequence  $e^{-n} \rightarrow 0$  as  $n \rightarrow \infty$ , so  $(e^{-n})$  is Cauchy in the usual sense, so  $(-n)$  is  $d_b$ -Cauchy. But suppose that  $(-n)$  converges to a real number  $x_0$  in  $(\mathbf{R}, d_b)$ . Then  $d_b(-n, x_0) \rightarrow 0$  as  $n \rightarrow \infty$ , which says that  $(e^{-n})$  converges to  $e^{x_0}$  in the usual metric on  $\mathbf{R}$ . But we know that  $(e^{-n})$  converges to 0 in the usual metric, so would have  $e^{x_0} = 0$ . But there is no such real number  $x_0$ , so  $(-n)$  cannot converge in  $(\mathbf{R}, d_b)$ . Hence  $(\mathbf{R}, d_b)$  is not complete.
- For similar reasons this is not complete either. Consider the sequence  $(n)$  in  $\mathbf{R}$ . The sequence  $(\tan^{-1}(n))$  converges to  $\frac{\pi}{2}$  in the usual sense as  $n \rightarrow \infty$ . So  $(\tan^{-1}(n))$  is Cauchy in the usual sense which says that  $(n)$  is  $d_c$ -Cauchy. But suppose that  $(n)$  converges to some  $x_0 \in \mathbf{R}$  in the metric space  $(\mathbf{R}, d_c)$ . Then  $(\tan^{-1}(n))$  converges to  $\tan^{-1}(x_0)$  in the usual sense. This gives  $\tan^{-1}(x_0) = \frac{\pi}{2}$ . But there is no such real number  $x_0$ . So  $(n)$  does not converge in  $(\mathbf{R}, d_c)$  and this space is not complete. ■

**Exercise 6.**

Show that the uniformly continuous image of a complete metric space is not necessarily complete.

*Solution.* Define  $f: [1, +\infty) \rightarrow (0, 1]$  by  $f(x) = \frac{1}{x}$ . ■

**\*Exercise 7.**

Suppose that  $f: X \rightarrow Y$  is a bijection between metric spaces such that  $f$  is uniformly continuous and  $f^{-1}$  is continuous.

- Prove that if  $(x_n)$  is a Cauchy sequence in  $X$ , then  $f(x_n)$  is a Cauchy sequence in  $Y$ .
- Prove that if  $y_n$  is a convergent sequence in  $Y$  then  $(f^{-1}(y_n))$  is a convergent sequence in  $X$ .
- Deduce that if  $Y$  is complete, then  $X$  is complete.
- Deduce that completeness is an invariant of uniform equivalence.

*Solution.* a. Let  $\varepsilon > 0$ . Then, by the uniform continuity of  $f$ , there is a  $\delta > 0$  such that  $d_X(x, y) < \delta \implies d_Y(f(x), f(y)) < \varepsilon$ . For this  $\delta$ , there is an  $N \in \mathbf{N}$ , such that  $m, n \geq N \implies d_X(x_m, x_n) < \delta$  since  $(x_n)$  is Cauchy. Therefore,  $m, n \geq N \implies d_Y(f(x_m), f(x_n)) < \varepsilon$ .

b. Follows from the (sequential) continuity of  $f^{-1}$ .

c. Let  $(x_n)$  be a Cauchy sequence in  $X$ . Then  $f(x_n)$  is a Cauchy sequence in  $Y$  by a which implies that  $f(x_n)$  converges in  $Y$  since we are assuming  $Y$  is complete. Therefore, by b,  $(f^{-1}(f(x_n))) = (x_n)$  converges in  $X$  and so  $X$  is complete.

d. If we suppose that  $f$  is now a uniform equivalence we immediately obtain that if  $Y$  is complete, then  $X$  is complete. By considering  $f^{-1}$  we also get the converse. ■

**Exercise 8.**

Let  $f: (0, \frac{1}{4}) \rightarrow (0, \frac{1}{4})$  be given by  $f(x) = x^2$ . Show that  $f$  is a contraction without a fixed point.

*Solution.* Firstly,  $f(0, \frac{1}{4}) \subseteq (0, \frac{1}{4})$ . Also,  $f$  is a contraction since

$$|f(x) - f(y)| = |x^2 - y^2| = (x + y) \cdot |x - y| < \frac{1}{2}|x - y|.$$

It has no fixed point since  $x^2 = x \implies x \in \{0, 1\}$ . ■

**Exercise 9.**

Let  $X = \{x, y, z\}$  and let  $d$  be the discrete metric on  $X$ . Define

$$\begin{aligned} d'(x, y) &= 2 = d'(x, z), & d'(y, z) &= 1, \\ d'(x, x) &= d'(y, y) = d'(z, z) &= 0. \end{aligned}$$

Show that  $d'$  is a metric on  $X$  that is Lipschitz equivalent to  $d$ . Define  $f: X \rightarrow X$  by

$$f(x) = y, \quad f(y) = z \quad \text{and} \quad f(z) = z.$$

Show that  $f$  is a  $d'$ -contraction but not a  $d$ -contraction.

*Solution.* It is easy to show that  $d'$  is a metric on  $X$ . To show that it is Lipschitz equivalent to  $d$  note that

$$\frac{1}{2}d'(a, b) \leq d(a, b) \leq d'(a, b) \quad \forall a, b \in X.$$

By putting  $K = \frac{1}{2}$  we can see that  $f$  is a  $d'$ -contraction by checking the definition. To show that  $f$  is not a  $d$ -contraction observe that

$$d(f(x), f(y)) = d(y, z) = 1 \leq Kd(x, y) = K \implies K \geq 1.$$

Since  $K$  has to be less than 1,  $f$  is not a  $d$ -contraction. ■

#### Exercise 10.

Let  $f: X \rightarrow X$  be a map of a compact metric space  $X$  such that

$$d(f(x), f(y)) < d(x, y)$$

for any distinct points  $x, y \in X$ . Prove that  $f$  has a unique fixed point.

*Hint:* Show that  $\inf\{d(x, f(x)) \mid x \in X\}$  is attained, and get a contradiction unless this inf is zero.

*Solution.* Consider the composition

$$d \circ (1 \times f) \circ \Delta: X \rightarrow X \times X \rightarrow X \times X \rightarrow \mathbf{R}.$$

Each map here is continuous so the composition,  $F$  say, is continuous. Since  $X$  is compact,  $F$  attains its lower bound  $l$ , say, on  $X$ . Suppose that  $l > 0$  and that it is attained at  $x_0$ , so  $d(x_0, f(x_0)) = l$ . Then  $x_0$  and  $f(x_0)$  are distinct, so by assumption  $l \leq F(f(x_0)) = d(f(x_0), f(f(x_0))) < d(x_0, f(x_0)) = l$ . This contradiction shows that  $l$  must be 0. Since  $l$  is attained  $d(p, f(p)) = 0$  for some  $p \in X$ , so  $f(p) = p$  and we have proved the existence of a fixed point.

Uniqueness: for distinct fixed points  $p, q$  we would have

$$d(p, q) = d(f(p), f(q)) < d(p, q),$$

a contradiction. So  $p$  is the unique fixed point. ■