The subgroup induction property

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14 April 2021

- Partially based on joint works with D. Francoeur and with R. Grigorchuk and T. Nagnibeda.
- Slides available at `www.leemann.website/slides/subgroupinduction.pdf`

Main goals

- Define a group property: *the subgroup induction property*,
- Show some interesting consequences of it,
- Exhibit groups with this property.

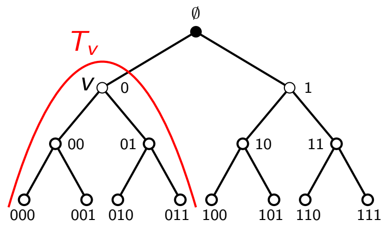
Advertisement

A property for subgroups of Aut(T), where T is a d-regular rooted tree. If G has the subgroup induction property, then, under some technical hypothesis,

- A full description of finitely generated subgroups of G,
- All maximal subgroups of G are of finite index,
- A nice description of weakly maximal subgroups of G (maximal among infinite index subgroups),
- G is torsion and just infinite,
- G is LERF (locally extensively residually finite),
- If H is a finitely generated subgroup of G, then H is commensurable with one of $\{1\}, G, \ldots, G^{d-1}$,
- If L is commensurable with G^n, then all its maximal subgroups are of finite index,
- Sub(G) has Cantor-Bendixon rank ω.

Regular rooted trees

- $T = T_d$: the d-regular rooted tree (the root has degree d and each other vertex has degree $d+1$);



- Vertices of T_d are in bijection with finite words on the alphabet $\{0, \ldots, d-1\}$ (root $\leftrightarrow \emptyset$ the empty word);
- The n^{th} level $\mathcal{L}_n$ of the tree is the set of vertices at distance n of the root;
- T_v is the subtree of T consisting of vertices below v.

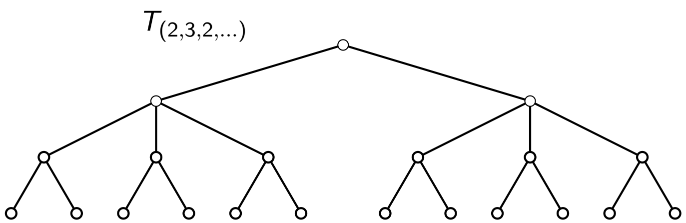
Identification of subtrees

Let T be a regular rooted tree and $v = x_1 \ldots x_n$ and $w = x'_1 \ldots x'_m$ be two vertices of T. Then

$$T_v = \{x_1 \ldots x_n y_1 \ldots y_k \mid k \in \mathbf{N}\}$$
$$T_w = \{x'_1 \ldots x'_m z_1 \ldots z_l \mid l \in \mathbf{N}\}$$

That is, we have a *canonical* isomorphism between T_v and T_w.

Spherically regular rooted tree



Let $(m_i)_{i \geq 0}$ be a sequence of integers greater than 1. One can define the corresponding spherically regular rooted tree $T_{(m_i)}$ as the rooted tree where every vertex of $\mathcal{L}_i$ has m_i children.

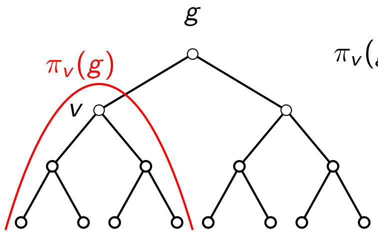
Importance of Aut(T).

Let $T = T_{(m_i)}$ be a spherically rooted tree.

- Any subgroup of Aut(T) is residually finite: $\bigcap_{[G:N]<\infty} N = \{1\}$,
- On the other hand, if G is a finitely generated residually finite group, then there exists T with $G \leq \text{Aut}(T)$.

Sections of elements of $\mathrm{Aut}(T)$

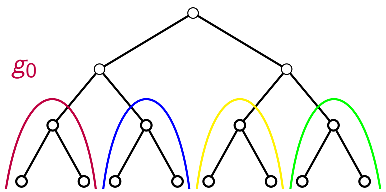
- For v a vertex of T and $g \in \mathrm{Stab}_{\mathrm{Aut}(T)}(v)$, the section $\pi_v(g) = g_{|_v}$ of g at v is the automorphism of T_v induced by g.



$$\pi_v(g) \in \mathrm{Aut}(T_v) \cong \mathrm{Aut}(T)$$

- Elements g that fixe $\mathcal{L}_n$ are usually described as the product of their sections:

$$g = (g_0, g_1, g_2, g_3)$$



Self-similar groups

Definition
A group $G \leq \mathrm{Aut}(T)$ is self-similar if for every vertex v in T we have $\pi_v(\mathrm{Stab}_G(v)) \leq G$.

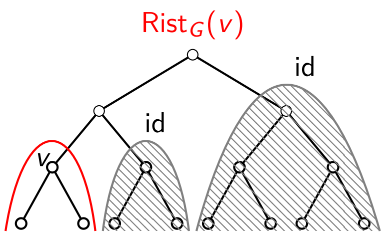
Definition
A group $G \leq \mathrm{Aut}(T)$ is self-replicating (or fractal) if for every vertex v in T we have $\pi_v(\mathrm{Stab}_G(v)) = G$.

Some subgroups of $\mathrm{Aut}(T_d)$

Let $G \leq \mathrm{Aut}(T_d)$. The following subgroups play an important role:

- Stabilizers of vertices $\mathrm{Stab}_G(v)$ and of rays $\mathrm{Stab}_G(\xi)$, $\xi \in \partial T$;
- Pointwise stabilizers of levels $\mathrm{Stab}_G(\mathcal{L}_n)$;
- Rigid stabilizer of vertices:

$$\mathrm{Rist}_G(v) := \{g \in G \mid g \text{ acts trivially outside } T_v\} = \bigcap_{w \notin T_v} \mathrm{Stab}_G(w)$$



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- Stabilizers of vertices $\mathrm{Stab}_G(v)$ and of rays $\mathrm{Stab}_G(\xi)$, $\xi \in \partial T$,
- Pointwise stabilizers of levels $\mathrm{Stab}_G(\mathcal{L}_n)$;
- Rigid stabilizer of vertices $\mathrm{Rist}_G(v)$,
- Rigid stabilizer of levels: $\mathrm{Rist}_G(\mathcal{L}_n) := \prod_{v \in \mathcal{L}_n} \mathrm{Rist}_G(v)$.
  Carefull: $\mathrm{Rist}_G(\mathcal{L}_n) \neq \mathrm{Rist}_{\mathrm{Aut}(T)}(\mathcal{L}_n) \cap G$.

The subgroup induction property (original definition)

Definition

Let $G \leq \mathrm{Aut}(T)$ be a self-similar group. A family $\mathcal{X}$ of subgroups of G is said to be *inductive* if

1. Both $\{1\}$ and G belong to X,
2. If $H \leq L$ are two subgroups of G with $[L : H]$ finite, then L is in $\mathcal{X}$ if and only if H is in $\mathcal{X}$,
3. If H is a finitely generated subgroup of $\mathrm{Stab}_G(\mathcal{L}_1)$ and all first level sections of H are in $\mathcal{X}$, then H is in $\mathcal{X}$.

Definition (Grigorchuk-Wilson, 2003)

A self-similar group G has the *subgroup induction property* if for any inductive class of subgroups $\mathcal{X}$, each finitely generated subgroup of G is contained in $\mathcal{X}$.

The subgroup induction property (alternative definition)

Definition (GLS, 2021)

A group $G \leq \mathrm{Aut}(T)$ has the *subgroup induction property* if for every finitely generated subgroup $H \leq G$, there exists n such that for every $v \in \mathcal{L}_n$, the section $\pi_v(\mathrm{Stab}_H(X))$ is either trivial or has finite index in $\pi_v(\mathrm{Stab}_G(X))$.

- G need not to be self-similar,
- For self-similar groups, the two definitions are equivalent [GLS, 2021],
- Examples: locally finite groups.

Branch groups: motivations

- Introduced in 1997 by Grigorchuk,
- Contain groups with unusual properties,
- Part of the classification of just infinite groups,
- Share some properties of $\mathrm{Aut}(T)$.

Branch groups

Definition

A subgroup G of $\mathrm{Aut}(T)$ is *branch* if for all n

1. G acts transitively on $\mathcal{L}_n$,
2. $\mathrm{Rist}_G(\mathcal{L}_n)$ is a finite index subgroup of G.

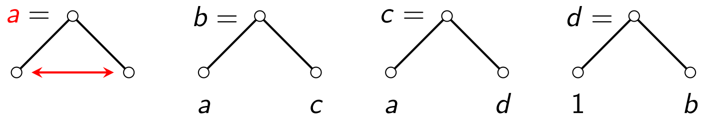
Example

The first Grigorchuk group $\mathfrak{G}$, the Gupta-Sidki p-groups ($p \geq 3$ prime), torsion `GGS` groups (acting on T_p, $p \geq 3$ prime).

All these examples are infinite, just infinite, torsion, of finite rank, all their maximal subgroups are of finite index, $\mathfrak{G}$ has intermediate growth, ...

The first Grigorchuk group

The first Grigorchuk group $\mathfrak{G} = \langle a, b, c, d \rangle$ acts on T_2 and is generated by



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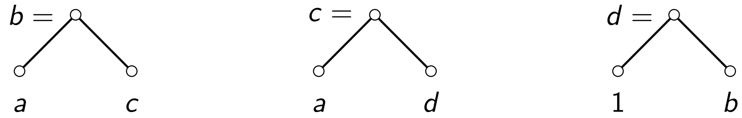

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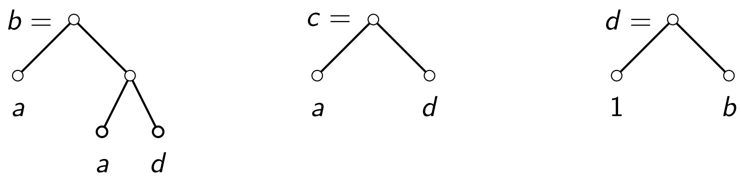

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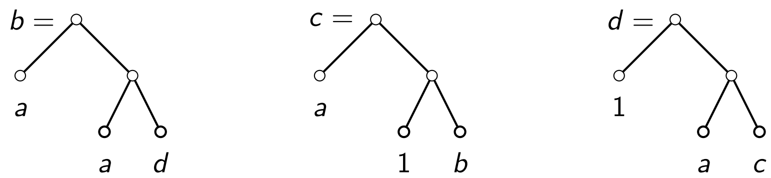
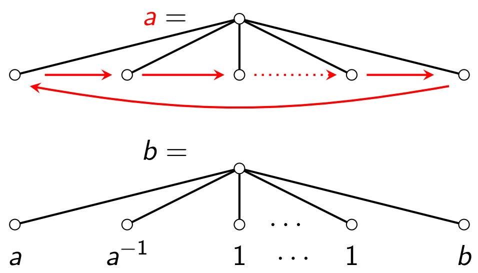


The Gupta-Sidki p-group

The group G_p acts on T_p ($p \geq 3$ prime) and is generated by a and b, where



GGS groups

- Let $p \geq 3$ be a prime and let $\mathbf{e} = (e_0, \dots, e_{p-2})$ be a vector in $(F_p)^{p-1} \setminus \{0\}$. The GGS group $G_{\mathbf{e}} = \langle a, b \rangle$ with defining vector $\mathbf{e}$ is the subgroup of $\mathrm{Aut}(T_p)$ generated by

  $a =$ cyclic permutation $(12 \dots p)$ of the first level vertices

  $b = (a^{e_0}, \dots, a^{e_{p-2}}, b)$,

- The group $G_{\mathbf{e}}$ is torsion if and only if $\sum_{i=0}^{p-2} e_i = 0$,
- The Gupta-Sidki p-group correspond to the special case $\mathbf{e} = (1, -1, 0, \dots, 0)$.

Properties

Let G be either the first Grigorchuk group, or a torsion GGS groups. Then

- G is self-replicating,
- $\mathrm{Stab}_G(v) = \mathrm{Stab}_G(\mathcal{L}_1)$ for every vertex v on the first level,
- G is a p group for some prime p,
- G is branch,
- G has the congruence subgroup property.

Groups with the subgroup induction property

Theorem (Grigorchuk-Wilson, 2003)

The first Grigorchuk group $\mathfrak{G}$ has the subgroup induction property.

Theorem (Garrido, 2016)

The Gupta-Sidki 3 group G_3 has the subgroup induction property.

Theorem (Francoeur-L, 2020)

The torsion GGS groups have the subgroup induction property.

Rough idea of the proof

Let $G = \langle a, b \rangle$ be a torsion GGS groups. For $g \in G$, define its *b*-length $|g|$ to be the minimum n such that $g = a^{i_1} b^{j_1} \dots a^{i_n} b^{j_n} a^{i_{n+1}}$ (a pseudo-norm).
Let $\mathcal{X}$ be an inductive classes of G and $H \leq G$ be a finitely generated subgroup. Then

- If there exists n such that $\varphi_v(\mathrm{Stab}_H(v)) \in \mathcal{X}$ for all $v \in \mathcal{L}_n$, then $H \in \mathcal{X}$,
- There exists $n = n(H)$ such that for all $v \in \mathcal{L}_n$ the subgroup $\varphi_v(\mathrm{Stab}_H(v))$ is generated by elements of b-length at most 1,
- If H is generated by elements of b-length at most 1, then $H \in \mathcal{X}$.

Some consequences (1)

Theorem (Francoeur-L, 2020)

Let G be a finitely generated branch group with the subgroup induction property. Then G is torsion, and hence just infinite (it is infinite and all its proper quotients are finite).

Theorem (F-L, 2020)

Let G be a finitely generated branch group with the subgroup induction property. Suppose that G is a p-group. Then all maximal subgroups of G are of finite index.

Some consequences (2)

Theorem (Gr-W;Ga;F-L)

Let $G \leq \mathrm{Aut}(T_d)$ be a self-replicating branch group such that $\mathrm{Stab}_G(v) = \mathrm{Stab}_G(\mathcal{L}_1)$ for every vertex v on the first level. Suppose that G has the subgroup induction property and let H be an infinite finitely generated subgroup of G. Then H is commensurable with one of $G, G^2, \dots, G^{d-1}$.
If moreover G is strongly self-replicating, has the congruence subgroup property and is a p-group, then all maximal subgroups of H are of finite index.

Some consequences (3)

- Any group G can be endowed with the profinite topology: the topology generated by finite index subgroups,
- G is residually finite iff $\{1\}$ is closed in the profinite topology,
- G is locally extended residually finite (LERF or subgroup separable) if all its finitely generated subgroups are closed in the profinite topology,
- $G \leq \mathrm{Aut}(T)$ has the congruence subgroup property if the profinite topology on G coincide with the $\mathrm{Aut}(T)$-topology.

Theorem (Grigorchuk-L-Nagnibeda,2020)

Let G be a finitely generated self-similar branch group with the congruence subgroup property and such that for every vertex v of the first level $\mathrm{Stab}_G(v) = \mathrm{Stab}_G(\mathcal{L}_1)$. If G has the subgroup induction property, it is LERF.

Strategy for the proof

1. Prove that the original definition is equivalent to the alternative definition,
2. Show a general result on subdirect products of just infinite groups,
3. Use it to have a nice characterization of finitely generated subgroups,
4. Conclude.

Classification of finitely generated subgroups

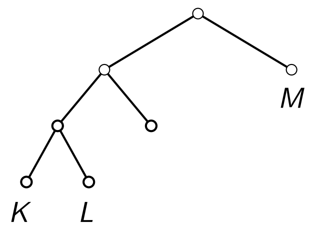
Theorem (Gr-L-N, 2021)

Let G be a finitely generated (self-similar) branch group such that $\mathrm{Stab}_G(v) = \mathrm{Stab}_G(\mathcal{L}_1)$ for every first level vertex v. Suppose that G has the subgroup induction property. If H is a finitely generated subgroup of G, there exists a block subgroup B of G with $B \leq H$ of finite index.

- In fact, finitely generated subgroups coincide with virtually block subgroups if and only if G has the subgroup induction property;
- But what are block subgroups?

Full block subgroups

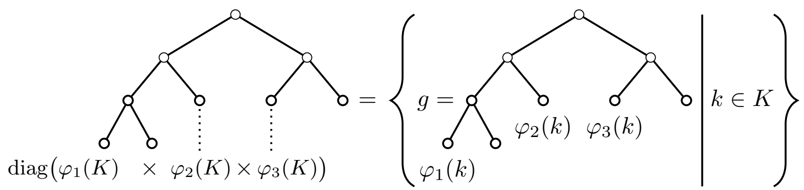
Let K be a finite index subgroup of G and v a vertex of T. Then one can define the group K_v of elements of $g \in \mathrm{Rist}(v)$ such that $g_{|_v} \in K$.



- K_v is naturally isomorphic to K,
- If G is self-similar, then K_v is a subgroup of G called a full block,
- If v and w are uncomparable, then $\langle K_v, L_w \rangle = K_v \times L_w$.

Diagonal subgroups

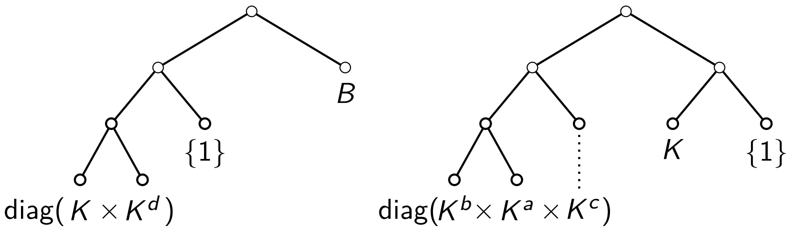
Let $v_1, \ldots, v_n$ be pairwise uncomparable vertices and $\varphi_1, \ldots, \varphi_n$ be automorphisms of K. This datas define a diagonal subgroup:



Block subgroups: definition

Definition
A block subgroup is a finite product of full blocks and of diagonal blocks (such that all the corresponding vertices are uncomparable).

Example



Block subgroups: properties

- If G is finitely generated, then block subgroups are finitely generated,
- If G is a branch group with the congruence subgroup property, then virtually block subgroups are closed in the profinite topology [L. 2020].

LERF

Theorem (G-L-N)
Let G be a finitely generated self-similar branch group such that $\mathrm{Stab}_G(v) = \mathrm{Stab}_G(\mathcal{L}_1)$ for every first level vertex v. Suppose that G has the subgroup induction property. Then finitely generated subgroups of G coincide with virtually block subgroups.

Corollary
Let G be as in the theorem. Suppose that G has also the congruence subgroup property. Then G is LERF.

The next step

We understand

- Finitely generated subgroups,
- Maximal subgroups.

The next step: understand weakly maximal subgroups.

Weakly maximal subgroups

Recall that a maximal subgroup of G is a maximal element in the lattice of proper subgroups of G.

Definition

A weakly maximal subgroup is a maximal element in the lattice of infinite index subgroups of G.

Weakly maximal subgroups

- If G is finitely generated, then every infinite index subgroup is contained in a weakly maximal subgroup (use Zorn's Lemma).
- If $M \leq G$ is both maximal and of infinite index, then it is weakly maximal.
- If $G \leq \mathrm{Aut}(T)$ is branch, then the parabolic subgroups $\mathrm{Stab}_G(\xi)$, $\xi \in \partial T$, are weakly maximal, infinite and pairwise distinct [Bartholdi – Grigorchuk, 2000].

Weakly maximal subgroups of branch groups

Question (Grigorchuk,2005)

Describe all weakly maximal subgroups of $\mathfrak{G}$.

- (Pervova, 2011) Concrete example of a weakly maximal subgroup W_P of $\mathfrak{G}$ which is not parabolic.
- (Bou-Rabee – L. – Nagnibeda, 2016) If G is branch and contains a finite subgroup F that fixes no rays, then it contains uncountably many non parabolic weakly maximal subgroups (non-constructive proof).
- (L., 2019) Complete description of the weakly maximal subgroups of $\mathfrak{G}$ and of torsion GGS groups.

Classification of weakly maximal subgroups

Theorem (L., 2019)

Let G be either the first Grigorchuk group, or a torsion GGS group. Weakly maximal subgroups of G are either generalized parabolic subgroups or virtually block subgroups. These two classes admit many characterization:

generalized parabolic	*virtually block*
finitely generated	*not finitely generated*
$\forall n \exists v \in \mathcal{L}_n : [\pi_v(\mathcal{G}) : \pi_v(W)]$ *is infinite*	$\exists n \forall v \in \mathcal{L}_n : [\pi_v(\mathcal{G}) : \pi_v(W)]$ *is finite*
$\forall v : \mathrm{Rist}_W(v)$ *is infinite*	$\exists v : \mathrm{Rist}_W(v) = \{1\}$
$W \curvearrowright \partial T$ *has infinitely many closed invariant subset*	$W \curvearrowright \partial T$ *has finitely many closed invariant subset*

Generalized parabolic subgroups

Definition

A generalized parabolic subgroup of $G \leq \mathrm{Aut}(T)$ is a setwise stabilizer $\mathrm{SStab}_G(C)$ where

- $C \subseteq \partial T$ is closed,
- C has empty interior (i.e. is nowhere dense),
- the action of $\mathrm{SStab}_G(C)$ on C is minimal.

Example

- Parabolic subgroups: $C = \{\xi\}$ for $\xi \in \partial T$,
- $C = F.\{\xi\}$ where F is a finite subgroup of G.

Generalized parabolic subgroups: properties

Lemma (L)

Let G be branch. Then generalized parabolic subgroups are infinite and pairwise distinct ($\mathrm{SStab}_G(C_1) \neq \mathrm{SStab}_G(C_2)$ if $C_1 \neq C_2$).

Corollary

Any branch group with an element of finite order contains a continuum of generalized parabolic subgroups that are not parabolic (they are all weakly maximal).

Block subgroups: properties

Let B be a block subgroup of a finitely generated, self-replicating branch group $G \leq \mathrm{Aut}(T)$. Then

- If B has no trivial blocks and at least one diagonal block, then it is of infinite index and every weakly maximal subgroup W containing B is not generalized parabolic,
- In particular, there exists infinitely many weakly maximal subgroups of G that are not generalized parabolic.

The next step (2)

We understand

- Finitely generated subgroups,
- Maximal subgroups,
- Weakly maximal subgroups.

The next step: understand the space $\mathrm{Sub}(G)$ of all subgroups of G.

The space Sub(G)

For a countable group G, there is a natural topology, the Chabauty topology on the set Sub(G) that turns it onto a totally disconnected compact topological space.
The Cantor-Bendixon rank of Sub(G) is the number of steps necessary to obtain a subspace of Sub(G) without isolated points:

- $X^0 := \mathrm{Sub}(G)$,
- $X^{\alpha+1}$ is X^α minus its isolated points,
- For λ a limit ordinal $X^\lambda := \bigcap_{\alpha<\lambda} X^\alpha$,
- The CB rank of Sub(G) is the least ordinal α such that $X^\alpha = X^{\alpha+1}$.

Subgroup induction property and the Cantor-Bendixon rank

Theorem (Wesolek-Skipper 2020; F-L)

Let G be a finitely generated regular branch group that is strongly self-replicating and such that for every vertex v of the first level, we have $\mathrm{Stab}_G(v) = \mathrm{Stab}_G(\mathcal{L}_1)$. Suppose that G has the congruence subgroup property and the subgroup induction property. Then Sub(G) *has Cantor-Bendixson rank ω.*

Corollary

The first Grigorchuk group as well as torsion GGS groups have Cantor-Bendixson rank ω.

The next step (3)

We understand

- Finitely generated subgroups,
- Maximal subgroups,
- Weakly maximal subgroups,
- The (Cantor-Bendixon rank of) the space Sub(G).

The next step: understand all subgroups of G that are closed in the profinite topology...