# Rigidity in Graphs and Groups

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My research interests lie in geometric and combinatorial group theory. I am particularly interested in groups acting on rooted trees, limits and coverings of (Schreier) graphs and rigidity phenomenon in Cayley and Schreier graphs.

The first thematic that interests me is the study of the so called branch groups. These are groups acting on a rooted tree that generalize the first Grigorchuk group, which was the first known example of a group of intermediate growth. The study of branch groups has been an active area of research these past years and many interesting results on their subgroup structures where obtained.

On the other hand, I am particularly interested in the notions of rigidity of (Cayley) graphs. One side of this question is to ask whenever a group $G$ admits a Cayley graphs with "few" automorphisms. The other, but related side, is to understand which Cayley graphs cover few other transitive graphs. These questions can be rephrased in a more group theoretic way that naturally leads to a strong (and geometric) version of the simplicity of a group.

Finally, I am also interested in de Bruijn's graphs and their generalizations: spider-web graphs and Rauzy's graphs associated to a subshift of finite type. These graphs naturally appears in many mathematical contexts (combinatorics, symbolic dynamic), but also in another areas (telecommunication, statistical physic, bio-informatic). The aim here is to compute the weak limit (also called Benjamini-Schramm) of these various graphs and to express it in terms of horospheric products of trees, such as the celebrated Diestel-Leader graphs. In a second time, we use the convergence to obtain more information on the limit, as for example the computation of the spectral zeta function. This problematic will not be developed further in the present project.

While the three above problematics are distinct, some common themes and technics are transversal to their study. It is for example the case of Schreier graphs and their geometry, of groups acting on trees, of limits of graphs, of just infinite groups as well as of the lattice of subgroups.

## 1 Branch Groups and their subgroups structure

Branch groups are groups acting on a rooted tree with "big" vertex stabilizers. They play an important role (together with simple groups and with hereditarily just infinite groups) in the classification of *just infinite* groups (infinite groups whose all proper quotients are finite) [9] and are hence of interest since every infinite finitely generated group admits a just infinite quotient. Self-similar groups appear naturally in holomorphic dynamics [24] and are examples of automata groups. Although quite different, these two classes of groups have large intersection, and many self-similar groups are also branch. In the class of finitely generated branch self-similar groups, there are torsion groups and torsion free groups; groups of intermediate growth and groups of exponential growth; amenable and nonamenable groups. Self-similar branch groups are a great source of examples and counter-examples in group theory and the study of these groups and their subgroup structure as attired a lot of attention among geometric group theorist these last years.

Formally, if $T$ is a $d$-regular rooted tree (the root has degree $d$ and every other vertex degree $d+1$), a subgroup $G \leq \mathrm{Aut}(T)$ is *branch* if the induced action of $G$ on $\partial T$ is minimal and if for every $n$ the subgroup

$$\prod_{v \text{ at distance } n \text{ from the root}} \mathrm{Rist}_G(v)$$

has finite index in $G$, where $\mathrm{Rist}_G(v) = \bigcap_{w \text{ is not a descendant of } v} \mathrm{Stab}_G(w)$ is the subgroup of element of $G$ that acts trivially outside $T_v$, the set of descendant of $v$.

Among popular examples of branch self-similar groups is the first Grigorchuk group $\mathcal{G}$ (which acts on the rooted binary tree). It was the first example of a group of intermediate growth (answering a question of Milnor, 1968), as well as of an amenable but not elementary amenable group (answering a question of Day, 1957) [7, 8]. Much is known about such subgroups of $\mathcal{G}$ as the stabilizers of vertices of the rooted binary tree (on which $\mathcal{G}$ acts) and of points on the boundary of the tree; the rigid stabilizers; the centralizers; subgroups of small index [11, 18] and maximal subgroups (which are all of finite index by [26]).

The study of branch self-similar groups is an active area of research, which has seen a lot of recent developments. I am particularly interested in the following two general themes. First to better understand $\mathcal{G}$ and its subgroup structure and secondly, to find families of groups that share some of the nice properties of $\mathcal{G}$.

In this context, the so-called *subgroup induction property* seems to play an important role. We will not give here a formal definition of this property, but rather focus on some of its consequences and of some of the open-problems related to it. It was used in particular to compute the Cantor-Bendixson rank of the set of subgroups of $\mathcal{G}$ [27]. In current projects with Francoeur and with Girgorchuk and Nagnibeda we also show the following

**Theorem 1** ([4, 12])**.** *Let $G$ be a self-similar branch group with the subgroup induction property, then*

1. *$G$ is of torsion and just infinite,*
2. *$G$ is* LERF *(*locally extensively residually finite[1]*, that is: every finitely generated subgroup is closed in the profinite topology),*
3. *The finitely generated subgroups of $G$ coincide with the subgroups with a block structure.*

Where a subgroup with a *block structure* is, up to finite index and roughly speaking, a product of copies of finite index subgroups of $G$, some of them embedded diagonally.

On one hand, Theorem 1 completely describes the finitely generated subgroups of $G$ while on the other hand it implies that $G$ is one of the few group that are known to be LERF, a property that is itself of great interest.

[1] Such groups are sometimes called *subgroups separable*.

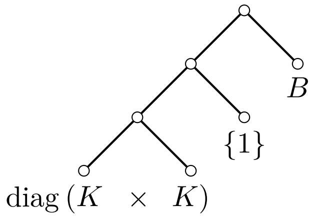


Figure 1: A block subgroup of the first Grigorchuk group.

The result about finitely generated subgroups of $G$ was first announced in [13] for the specific case of the first Grigorchuk group. In this case, Grigorchuk and Nagnibeda announced moreover the existence of an algorithm that given a finite subset $S$ of $G$ returns the block structure of $\langle S\rangle$. It is unclear if such an algorithm exists in the general case.

**Question 2.** *Let $G$ be a self-similar branch group with the subgroup induction property. Does it exist an algorithm that given a finite subset $S$ of $G$ returns the block structure of $\langle S\rangle$?*

On the other hand, since maximal subgroups, as well as other subgroups of small index, of $\mathcal{G}$ are well understood, it is natural to try to understand *weakly maximal subgroups* of $\mathcal{G}$. These are the subgroups of infinite index that are maximal with respect to this property. Indeed, from the point of view of geometric group theory, finite index subgroups look like $G$ and are, in some sense, not interesting. It is therefore normal to turn It was asked in [10] if these were the only examples.

**Question 3** ([10])**.** *Describe all weakly maximal subgroups of $\mathcal{G}$.*

In [1], together with Bou-Rabee and Nagnibeda, we proved that every branch group (subject to some minor technical condition) contains uncountably many weakly maximal subgroups that are distinct from stabilizers of rays. In a recent work, I was able to give a full description of weakly maximal subgroups of self-similar branch groups with the subgroup induction property, hence answering Question 3. They either have a block structure or are *generalized parabolic subgroups*, that is the setwise stabilizer $\mathrm{Stab}_G(C)$ of a closed but not clopen subset $C$ of $\partial T$ such that $\mathrm{Stab}_G(C)$ acts minimally on $C$.

**Theorem 4** ([20])**.** *Let $G$ be a self-similar branch group with the subgroup induction property The weakly maximal subgroups of $G$ split into two classes: generalized parabolic subgroups and weakly maximal subgroups with a block structure. These two classes admit many characterizations, as shown in table 1.*

In regards to the above theorem, the following remains open

**Question 5.** *Let $G$ be a self-similar branch group with the subgroup induction property*

1. *Is it possible to describe closed but not clopen subset $C$ of $\partial T$ such that $\mathrm{Stab}_G(C)$ acts minimally on $C$?*

| generalized parabolic | weakly maximal with a block structure |
|---|---|
| not finitely generated | finitely generated |
| $\forall v : \mathrm{Rist}_W(v)$ is infinite | $\exists v : \mathrm{Rist}_W(v) = \{1\}$ |
| $W \curvearrowright \partial T$ has infinitely many orbit-closures | $W \curvearrowright \partial T$ has finitely many orbit-closures |
| $\forall n \exists v \in \mathcal{L}_n : [\pi_v(G) : \pi_v(W)]$ is infinite | $\exists n \forall v \in \mathcal{L}_n : [\pi_v(G) : \pi_v(W)]$ is finite |

Table 1: The two classes of weakly maximal subgroups of $\mathcal{G}$. Here $\pi_v(G) = \pi_v(\mathrm{Stab}_G(v))$ is the projection (also called section) of elements of $\mathrm{Stab}_G(v)$ on the automorphism group of the subtree rooted at $v$.

2. *Is it possible to describe which block structures occur in weakly maximal subgroups of $G$?*

As we have just seen, the induction subgroup property happens to be a really powerful tool. At this point, it is natural to ask

**Question 6.** *Which branch self-similar groups do have the subgroup induction property?*

As for now, the only known-examples of group with the subgroup induction property are the first Grigorchuk group [11] and the Gupta–Sidki 3-group [5]. I plan to show, using the technics developed in [11, 5, 20], that all the torsion *GGS groups* have the subgroup induction property. When this will be done, I plan to look at the more general family of torsion *generalized multi-edge spinal groups.*

Torsion generalized multi-edge spinal groups (and hence torsion GGS groups) are known to share a lot of the nice properties of the first Grigorchuk group. They are torsion, just infinite, regular branch, self-replicating (fractal) and hence self-similar, have the congruence subgroup property, do not have maximal subgroup of infinite index... The proofs of the subgroup induction property for the first Grigorchuk group and the Gupta–Sidki 3-group suggest that these properties are not independent.

**Question 7.** *Let $G$ be a self-similar branch group. What is the relation between the subgroup induction property and*

1. *the congruence subgroup property?*
2. *having no maximal subgroup of infinite index?*

The subgroup induction property can be generalized [12] to any group of homeomorphisms of a Cantor space, but in this context it is reasonable to ask that $G$ is self-similar.

**Question 8.**
1. *Which of the above results can be generalized for self-similar groups?*
2. *Can the subgroup induction property been read on the automaton defining $G$?*

On the other hand, I would like to investigate (system) of equations in branch groups, for example the local to global property. I strongly believe that the subgroup induction property and the well-understanding of the profinite completion are important tools that may been used to generalize results known for free groups, as for example [2].

## 2 Graphs: transitivity, rigidity and simplicity

Given a base graph, the structure of its covers is well known, notably for finite degree covers: algorithm, existence [22], and more. However, the inverse problem — given a graph $\Gamma$, find all the graphs that are covered by $\Gamma$ — is more difficult. This is one of the problems I will discuss here. We will also be interested at the transitivity (on the vertices) of a graph, at simple groups and at Cayley graphs with few automorphisms.

The study of the connectivity constant leads Benjamini to formulate the following

**Conjecture 9** (Benjamini)**.** *There exists a constant $M$ such that every infinite transitive graph (not quasi-isometric to $\mathbf{Z}$) covers an infinite transitive graph with girth*[2] *at most $M$.*

A positive answer to this question will allow to restrict in some case (for property that may be "lifted up") the study of transitive graphs to the one with small girth.

Not only the answer to this question is not known, but even the following form remains open.

**Conjecture 10.** *Every infinite Cayley graph (not quasi-isometric to $\mathbf{Z}$) covers another infinite transitive graph.*

In a first step, we can restrict ourselves to coverings that are compatible with the labelling of the Cayley graph[3]. This version is more natural from an algebraic point of view due to the following fact. An infinite Cayley graph of $G$ does not cover any another infinite Cayley graph by a covering compatible with the labelling if and only if $G$ is just infinite.

In this restricted setting, I gave in [19] a negative answer to Conjecture 10. This is done in two steps. Firstly, given a free group $F$ and a free generating set $S$, I gave a characterization of subgroups $H$ of $F$ such that the corresponding Schreier graphs are transitive. Then I used this characterization to show

**Theorem 11** ([19])**.** *Let $\Gamma$ be Cayley graph of a Tarski's monster and $\varphi\colon \Gamma \to \Delta$ be a covering that is compatible with the labelling. Then either $\Delta = \Gamma$, or $\Delta$ is reduced to one vertex, or $\Delta$ is not transitive.*

Counterexamples to the labelled version of Conjecture 10 are groups that are strongly, in a geometric way, just infinite. Tarski's monsters are even *strongly simple* in the following sense. Recall that a group $G$ is simple if and only if any of its Cayley graphs does not cover, by labelling compatible covering, another Cayley graph. We say that $G$ is *strongly simple* if it admits a Cayley graph that does not cover, by labelling compatible covering, another transitive graph. This definition allows us to "quantify" the simplicity of a group and may be useful to better understand the structure of infinite simple groups. An important remark at this point is that they both exists strongly simple groups (Tarski's monsters) as well as simple groups that are not strongly simple (the finite alternate groups) [19].

I plan to work on the following open questions

[2] The *girth* of a graph is the length of the shortest cycle.

[3] A covering $\varphi\colon \mathrm{Cayley}(G, S) \to \Gamma$ is *compatible with the labelling* if for any edge $e$ of $\Gamma$, all its preimages by $\varphi$ have the same label.

**Question 12.** *1. Does there exist finite strongly simple groups that are not cyclic?*

*2. Does there exist infinite simple groups that are not strongly simple?*

*3. Does there exist strongly simple infinite groups of finite presentation?*

Currently, the methods of [19] work only for subgroups of prime order, but it should be possible to extend them to take care of all subgroups that are $p$-groups.

I also plan to pursue the work began in [19] on the structure of the set of *transitive subgroups* (that is subgroups with a transitive Schreier graph) of $G$. I proven that this set is stable by intersection, which raises the following

**Question 13.** *Let $A$ and $B$ be two subgroups of $G$ such that their respective Schreier graphs are transitive. Is the Schreier graph of $\langle A, B\rangle$ transitive?*

If the answer is positive, it implies that the set of transitive subgroups is a sublattice of the lattice of subgroups of $G$. In this case, the study of this sublattice and of these properties should be of great help to answer the above questions. Speaking of lattice, it is interesting to observe that all the following properties admit a lattice-theoretic characterization: being simple, being a Tarski monster, being a finite index subgroup, being a normal subgroup. We can thus ask

**Question 14.** *Does the fact to be strongly simple admits a lattice-theoretic characterization?*

On the other hand, and as remarked in [23], it could be interesting to generalize the methods of [19] to the case of simplicial complexes.

In a recent work with de la Salle [21] we proved that if $G$ is a Tarski's monster, then it admits a Cayley graph $\Gamma$ such that every covering $\varphi\colon \Gamma \to \Delta$ that is bijective on the balls of radius 1 preserves the labelling. When combined with the results from [19], we obtain that the graph $\Gamma$ does not covers any other infinite transitive graphs, except maybe by coverings that are not bijective on the balls of radius 1. In order to give a definitive answer to Conjecture 10, it remains to study coverings $\varphi\colon \Gamma \to \Delta$ that are not bijective on the ball of radius 1.

The above result on Tarski's monsters follows from a rigidity criterion for Cayley graphs which is itself of great interest and whose story goes back to the 1970s. A Cayley graph $\Gamma$ of a group $G$ is said to be a *graphical rigid representation* (or *GRR*) if the only automorphisms of $\Gamma$ are the one given by the left multiplication of elements of $G$. Such a graph has as few automorphisms as possible while still being a Cayley graph. Abelian groups of exponent greater than 2 cannot have a GRR (take the inverse function), and the same is true for the so-called generalized dicyclic groups.

**Conjecture 15** (Watkins [31])**.** *There is an integer $n$ such that every group of cardinality at least $n$ and which is neither abelian of exponent greater than 2 nor generalized dicyclic admits a GRR.*

Several cases of Conjecture 2 have been known for a long time. Most importantly, it has been known from the 1970s for finite groups, thanks to combined efforts of, notably, Imrich, Watkins, Nowitz, Hetzel and Godsil, [15, 3, 28, 25, 29, 30, 16, 17, 14, 6]. Moreover the finite exceptional groups are completely understood: there are 13, all of whom of cardinality at most 32, see [6] and the references therein. On the other hand, Watkins showed [31] that the conjecture holds for free products of groups. Finally, with de la Salle we recently proved

**Theorem 16** ([21])**.** *Let $G$ be a finitely generated group that is neither abelian nor generalized dicyclic. If $G$ has an element of order at least $\big(2\operatorname{rank}(G)\big)^{36}$ then it admits a GRR.*

Therefore, for finitely generated groups, the conjecture remains open only in the class of torsion groups with bounded torsion. Preliminary results allowed us to narrow the class of possible counterexample to Conjecture 2 to such groups satisfying some additional restrictive conditions. These conditions depend on the size of centralizers of elements and on the number of square roots of a given element. We plan to further elaborate the methods of [21] in order to get ride of the last possible counterexamples and hence give a final answer to Conjecture 2. In [21] we also obtained results about the oriented version of Conjecture , namely that

**Theorem 17** ([21])**.** *Let $G$ be a finitely generated group. If $G$ has an element of order at least $\big(5\operatorname{rank}(G)\big)^{12}$ then it admits a ORR (an oriented Cayley graph $\vec{\Gamma}$ with $G = \operatorname{Aut}(\vec{\Gamma})$).*

More precisely, Theorem follows from Theorem and from another result of [21]: a characterization of abelian and generalized dicyclic groups in terms of the automorphism group of their Cayley graphs.

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